

Additions in Arithmetic, 1483-1700, to the
Sources of Cajori's
"History of Mathematical Notations"
and Tropfke's
"Geschichte Der Elementar-Mathematik"

BY

SISTER MARY LEONTIUS SCHULTE

of the

Sisters of Saint Francis, Rochester, Minnesota

A DISSERTATION

SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF DOCTOR OF PHILOSOPHY
IN THE UNIVERSITY OF MICHIGAN

1934

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Ann Arbor, Michigan
Feast of the Sacred Heart,
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I

INTRODUCTORY CHAPTER

A. Purpose of this Study

It will be the purpose of the present study, after pointing out that the standard works on arithmetical notations are demonstrably incomplete, to supplement these works with the results of further research, particularly among the arithmetics of the sixteenth and seventeenth centuries. The new material should complete Cajori's A History of Mathematical Notations¹, v. I, 1928, and certain sections of Tropicke's Geschichte der Elementar Mathematik², v. I, II, III, 2nd. ed., 1921, not only by noting the mathematical notations of writers who have been given little or no mention in these works, but also by pointing out phases of the treatment of the arithmetic material which have been passed over in silence by these authors. In particular, the attempt has been made to increase the source material of a period when notations were undergoing rapid change, the sixteenth and early seventeenth centuries. These studies have been undertaken with a view to increasing the usefulness of these works to teachers of elementary mathematics, to historians of this field, to historians in general, and to students of mathematics.

There is no questioning the fact that Cajori's Notations, Tropicke's Geschichte, and D. E. Smith's Rara Arithmetica³ (Boston, 1908) are the fundamental works on the history of arithmetic after the invention of printing. The first two are widely cited and used; yet they present only a selected list of authors, without reference either to the number of editions or to other indications of the influence of the given works. Even when authors not cited by Cajori or by Tropicke use notations that are the same as, or similar to, the notations of the authors who are cited by them, it is highly desirable to have this fact noted as a matter of record. In the library at the University of Michigan there is a collection which permits this extension. Some

1. Hereafter referred to as Cajori, Notations.

2. Hereafter referred to as Tropicke, Geschichte. Several references are to v. I, II, 3d. ed., 1930, 1933.

3. Hereafter referred to as D. E. Smith, Rara.

one hundred works have been investigated. Of these, some merit more attention than has been given them by the above historians, others have not been treated at all. Further than this, there are numerous different editions of the works cited by both Cajori and Tropfke which have been examined and which have been found to contain variations worthy of note. With the resources of the University of Michigan supplemented by a study in the libraries of New York City, the Library of George A. Plimpton, the Library of Columbia University, the Library of David Eugene Smith (Teachers College, Columbia University), and the New York Library, the writer has been able to add materially to the sources utilized by the standard works on mathematical notations.

For the completion of material given by Cajori, it seems desirable to use the method employed by him, having a preliminary discussion of the important individual writers followed by a detailed study of notations involving not only these individual writers but numerous others. It has been deemed best to separate the topical surveys of Cajori and Tropfke, supplementing the former's material on notations and supplementing the latter's material on methods. It is believed that in this parallel study of the two works by Cajori and Tropfke, the limitations and also the strong points of both works will be emphasized.

The particular group of writers selected for more detailed study (No. 1, 19, 29, 32, 33, 37, 47, 52, 54, 56, 63, 67, 74, 82, 85, 89, 91, 100) has been chosen chiefly because of numerous indications of their relatively greater importance. This is evidenced by the number of editions of their works, by translations, and by the use of their material by other writers. Ten of this group of writers produced works of at least five editions, and yet seven are not cited by Cajori and eight not by Tropfke. In fact, the textbooks in arithmetic of seven writers of this group are 'centenarians'¹, but only two of their authors, Gemma Frisius and Oronce Finé, are cited by both Cajori and Tropfke. The authors of the remaining five 'centenarians', Hemeling in Germany, Feliciano, Lapazzaia, and Figatelli in Italy, and Savonne in France, produced works which were favorably received by their contemporaries, if one makes his judgment on the basis of their circulation over this long period of

1. Textbooks with a long life. See account given by L. C. Karpinski, "Arithmetic Centenarians", *Scripta Mathematica*, Nov. 1933, pp. 34-40. These textbooks continued to be published over a period of one hundred years.

time. The influence of these 'centenarians' can hardly be disputed.

Of the numerous further sources not cited by either Cajori or Tropicke, some works were widely used which may have had limited editions, as was the work of Leonard of Pisa to whom numerous references are made by later writers. We have in mind also such authors as Cataneo, Peverone, Bassi, and Ghaligai.

It is difficult to ascertain what principles Cajori used in his selection of authors. A number of the writers drawn upon in the following pages have been chosen because of the continued use of their works over a long period of time, as noted above. The works of such authors as Tagliente, Hodder, and Reisch, though not 'centenarians', passed through many editions; and yet these are cited by neither Cajori nor Tropicke. Some authors such as Feliciano, Hemeling, Köbel, Albert, Finé, and Gemma Frisius deserve more than their one reference or at most three references in Cajori. Again, for certain authors (Clavius, Tartaglia, Cardan, Bombelli, Vieta, Descartes, Girard, Harriot, Rudolff, Rahn, and Scheubel) who are included in Cajori, particularly important further notational references are given. The including of still others may have its justification in the fact that their works continued to be published a century or more after the appearance of the first edition; for example, Bassi and Hemeling. There are also those works which are worthy of addition because of their wide use in translations in other countries; the work of Ortega is a case in point.

For the most part, references are made to the peculiarities in symbolism not noted by Cajori and to the treatment of arithmetic material not noted by Tropicke in works rich in notation and relatively influential. For instance, in the prefaces of many works may be found references to the influence of earlier or contemporary writers. This is the case in the arithmetic of the Italian, Lorenzo Forestani (No. 35.), in which among the writers included are such names as Tartaglia, Ghaligai, Sfortunati, Cataneo, Lapazzala, Ortega, Clavius, and Peverone, all listed in the present bibliography, but of whom only two are mentioned in Tropicke. About the same time, Hulsius, in his Tractatus Primus Instrumentorum (Frankfurt, 1604, U. of M.), makes reference to those writers: Peurbach, Gemma Frisius, Clavius, Ghaligai, Cataneo, and Waser, of whom the last three are not found in Tropicke. In an earlier work, Antich Rocha¹ in his

1. D. E. Smith, *Rara*, p. 316.

Arithmetica (Barcelona, 1565) has likewise consulted, among others, the authors Scheubel, Finé, Ramus, and Gemma Frisius. Again, the work of Graffenried (No. 42.), although it itself is not so well known, embodies the labors of eminent predecessors, such as Cardan, Clavius, Rudolff, Gemma Frisius, Glareanus, Alsted, Ramus, and Jacob.

A modern writer on notations, J. W. L. Glaisher¹, mentions these early Italian authors: Borgi, Calandri, Pellos, Pacioli, Tagliente, Feliciano, Sfortunati, Cataneo, Ghaligai, and Tartaglia. Of these, Calandri, Tagliente, and Cataneo are not noted in Cajori, while Pellos and Feliciano are given only one reference. It is important, then, to have a more complete list of sources, one which will include these writers who are of almost equal prominence with Borghi and Pacioli.

Another contemporary writer, J. Pelsener² (Brussels), states in Isis that it would be interesting to illustrate by new examples the conclusions of Cajori in his Mathematical Notations. His brief note brings to light a point which is in danger of being obscured by the mass of data in Cajori, namely that there was a great body of mathematical symbols in evidence at the end of the sixteenth century and in the seventeenth century which had a wide circulation--especially in the works of the secondary authors.

Finally, it should be noted that Cajori dismisses the important topic of finger and counter reckoning in a very few lines, while this material was widely used and had considerable influence upon arithmetical terminology.

The desirability of this study finds its justification in the fundamental importance of Cajori's treatise as a work of reference. The editor of Isis himself, George Sarton³, reviews it thus:

Cajori's History of Mathematical Notations will remain a standard work for many years to come.

Speaking of Cajori and his work, R. C. Archibald⁴ writes:

1. J.W.L. Glaisher, "On the Early History of Signs + and - and on the Early German Arithmeticians", Messenger of Mathematics, v. 51 (1921), pp. 77, 78.
2. J. Pelsener, "A propos de notations mathematiques", Isis, v. 17, 1932, p. 259.
3. G. A. Sarton, Review in Isis, v. 13, 1929, pp. 129, 130.
4. R. C. Archibald, "Florian Cajori 1859-1930", Isis, v. 17, 1932, p. 386.

His monumental work on Mathematical Notations, original in conception, must be of enduring value.

In a similar way, Tropfke's Geschichte is also so important as a work of reference that a discussion of the sources is highly desirable.¹

B. Tabular View of Additions to Our Knowledge of Notations Not Mentioned by Cajori or by Tropfke

In view of the fact that one-half of the authors listed in the present bibliography are not included by Cajori or by Tropfke considered together, and since approximately two-thirds and almost three-fourths are omitted by Cajori and Tropfke respectively, it is desirable to complete their lists of sources. It is of paramount importance to note that a very large number of the writers of the sixteenth century given by D. E. Smith in his Rara Arithmetica is missing entirely in Cajori. Incidentally, too, I have made references to some works not cited in D. E. Smith's very complete bibliography, but available in the University of Michigan Library and the G. A. Plimpton Library in New York (No. 26, 30, 31.).

A true estimate of D. E. Smith's bibliography has been given by L. C. Karpinski² when he writes:

Bibliographically the Rara Arithmetica will always be an authority in so far as arithmetical books of the period are concerned.

In this work D. E. Smith has amassed a total of over four hundred and fifty authors who wrote arithmetics in the fifteenth and sixteenth centuries. Including editions, there are more than one thousand different items. Of these fifteenth and sixteenth century authors, Mr. Plimpton's library includes a majority, and certainly the more important ones. Cajori and Tropfke have used less than one-half of the writers in Dr. Smith's bibliography. The present study involves the study of an appreciable further group of the

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1. See the review of Tropfke's work, Bibliotheca Mathematica, 3d. ed., v. 4, 1903, pp. 213-218. Also Archeion, v. 13, 1931, pp. 102-3; v. 15, 1933, p. 456.
 2. Karpinski, "A Unique Collection of Arithmetics", Popular Science Monthly, Sept. 1910, pp. 226-233.

sixteenth-century authors, making it far more inclusive of the sixteenth-century works available in American libraries.

1. Authors not mentioned by Cajori or Tropfke.
(Numbers refer to the bibliography)

4, 5, 6, 7, 8, 9, 12, 15, 19, 21, 22, 23, 26, 27, 30, 31, 32, 35, 36, 41, 42, 43, 44, 46, 50, 51, 53, 55, 56, 57, 58, 59, 60, 61, 62, 66, 68, 69, 72, 77, 80, 82, 84, 87, 88, 89, 92, 94, 97, 98, 99, 100, 101.

2. Authors not mentioned by Cajori.

4, 5, 6, 7, 8, 9, 10, 12, 13, 14, 15, 19, 20, 21, 22, 23, 26, 27, 30, 31, 32, 35, 36, 40, 41, 42, 43, 44, 46, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 66, 68, 69, 71, 72, 77, 80, 82, 84, 87, 88, 89, 92, 94, 97, 98, 99, 100, 101.

3. Authors not mentioned by Tropfke.

1, 3, 4, 5, 6, 7, 8, 9, 12, 15, 19, 21, 22, 23, 26, 27, 29, 30, 31, 32, 35, 36, 38, 41, 42, 43, 44, 46, 47, 48, 50, 51, 53, 55, 56, 57, 58, 59, 60, 61, 62, 63, 65, 66, 68, 69, 72, 76, 77, 80, 82, 84, 85, 86, 87, 88, 89, 91, 92, 93, 94, 97, 98, 99, 100, 101.

II

COMPLETION OF CAJORI'S NOTATIONS

A. Notations Used by Individual Writers1. German: Johannes Husswirt (No. 52.)
(1501)

Husswirt's Enchiridion, written in Latin, bears in the colophon the spelling huswirt, with the date MCCCCI (f. 20, r.). This book is not mentioned by Cajori, although it would appear to be noteworthy because it had five subsequent editions, and was published with historical notes by Professor Wildermuth, at Tübingen, in 1865. De Morgan¹ calls it 'a short treatise, apparently one of the earliest printed in Germany on the Arabic system.' It is an evidence of the fact that chapters on doubling and halving persisted well into the sixteenth century.

The number 1 is like a printed j, and as D. E. Smith² remarks, Husswirt gives four names to the tenth character, 0, which still has no generally accepted name in the English language:

- Decima ~~X~~0 theca. circul⁹. cifra. sive figure nihili appellat.

These four names appear in Sacrobosco's Algorismus. Husswirt recommends the regular form X in checking results by casting out the 9's and 7's, although he uses it but twice (ff. 4, r., 10, r.). These 'due linee oblique descendentes ac se intersecantes per modum crucis hoc modo (quali) X' (f. 3, r.) are dispensed with in all other cases.

Husswirt performs subtraction and multiplication by the complementary method (ff. 4, r., 4, v.).³ In division, the divisor is below the dividend, while the quotient is on the right (f. 6, r.). Duplation follows multiplication,

-
1. Augustus De Morgan, Arithmetical Books from the Invention of Printing to the Present Time, (London, 1847), p. 3.
 2. D. E. Smith, Rara, p. 76.
 3. Dr. Susan R. Benedict, A Comparative Study of the Early Treatises Introducing into Europe the Hindu Art of Reckoning, p. 73, says that complementary multiplication appears in the works of the thirteenth century.

and mediation, division; but in dealing with counters (*proiectilibus*) and fractions he places them before multiplication, because they are needed there in abacus calculating.

A dot between the terms of both arithmetic and geometric progressions is his only symbol (f. 7, v.).

The terms of a fraction are widely separated by a fractional line, broken in the case of two or more digit numbers, as was general in the early days of printing (f. 12, r.).

2. German: Johannes Wolphius (No. 100.) (1534)

The Rudimenta Arithmetices was first printed in Latin at Nürnberg in 1527; this study concerns the edition of 1534, Frankfurt. The text of Wolphius is very similar to that of Husswirt. Twenty-seven folios are devoted to arithmetic, while the remaining twenty-nine contain elements of geometry by Johannes Vogelín. At the end is the date MDXXXIIII (f. 56, r.).

Wolphius calls 0 cifra or nulla (f. 3, r.). In grouping digits in numeration he writes 1545416 (f. 3, v.). The check by nines is not mentioned by him; seven is used throughout.

Duplation and mediation respectively precede multiplication and division, which order is different from that used by Husswirt. A short method of dividing by 100 is an additional feature; Wolphius divides 63740 by 100 thus:

637|40

1|00 (f. 9, v.). When reducing fractions to lowest terms, vertical bars take the place of the equality sign (f. 16, v.).

3. Dutch: Gemma Reinerus, Frisius (No. 37.) (1549, 1550)

The Arithmeticae practicae Methodus Facilis Per Gemmam Frisium, Medicum Ac Mathematicum is the beginning of the title of an arithmetic the most popular in the Latin schools of the sixteenth century¹, as evidenced by fifty-nine or more editions in that century cited by Smith, besides numerous editions after 1600.² The very fact that it was in use

1. D. E. Smith, Rara, p. 201.

2. The editions we use, 1549 and 1550, were printed at Paris and are substantially the same. All references are to the 1549 ed.

for more than a century indicates its degree of appeal to teachers during that period. The arithmetic of Gemma Frisius was prescribed in the curriculum of the seventeenth century.¹

There is little in Gemma's arithmetic to distinguish it from the early commercial arithmetic of the Italian writers. Gemma calls zero cyphram, and says it is written like the letter q, or a circle (f. 3, v.). Vertical bars serve to group the digits in numeration, as 23|456|345|678 (f. 4, v.). He uses guiding lines in multiplication by the complementary method (f. 4, r.), in reduction of fractions to a common denominator (f. 22, v) as well as in application of the rule of false position (f. 36, r.), but he fails to use them in casting out 9's.

Gemma Frisius finds it convenient in multiplication and division to cancel the digits in the multiplier and dividend as soon as used. There is a section on multiplication by 10 and 100 (f. 11, r.) followed by a discussion on duplation and mediation. Because he does not consider these latter as distinct operations from multiplication and division, they are not treated separately as was customary in the Latin books of that time.

The terms of a progression are separated by a comma; progressions, however, are no longer included by Gemma among the fundamental operations. In extracting roots of numbers, grouping is effected by dots below, thus: 1190 $\dot{2}$ $\dot{5}$ (f. 41, r.).

4. Italian: Girolamo Tagliente (No. 89.) (1520)

An author whose work was so widely used that it probably had thirty-six editions should be important enough to be included among the individual writers listed by Cajori in his History of Mathematical Notations. Such was the popular mercantile treatise on arithmetic published in Italian by Tagliente.

This textbook was influential in spite of the arrangement of the fundamental operations, in which multiplication and division precede addition and subtraction. Curious also are his forms of multiplication (ff. 9, v., 11, r., 11, v.).²

1. Friedrich Grundle, Die Mathematik an den Deutschen Höheren Schulen, (Berlin, 1928), p. 95, 107. The author singles out the arithmetic of Gemma Frisius as a representative text of this century and discusses it at length, pp. 75-80.

2. Two forms are illustrated in Smith's Rara, p. 116.

Tagliente introduces his book with finger symbols (ff. 3, v., 4, r.), also used in a later edition.¹ He follows Calandri in accompanying problems with illustrations.

Instead of the regular form X in the proof of 9's, Tagliente uses the Greek cross (f. 13, v.). Division of 3497 by 100 he expresses thus: $34|97$ (f. 16, r.), as did Wolphius a few years later.

Tagliente's work is truly a representative Italian arithmetic of the early sixteenth century², for it enters into the various minute details of the operations and commercial rules of the Italian arithmeticians. For example, Tagliente gives several methods for the multiplication of integers, and he deals chiefly with denominate numbers.

5. Italian: Pietro Cataneo (No. 19.) (1546, 1559)

Although the work of Cataneo was greatly influenced by his Italian predecessors, it introduces several peculiarities in notation. We draw our conclusions about this work from a very rare edition, his first (Venice, 1546), and further give indication of the use of notations in his second edition (1559).

Cataneo uses Zero for 0; and following Leonardo of Pisa and other early writers, he arranges the nine digits in decreasing order (f. 2, r.). In his first edition dots enclose all integers in the running text, whereas in that of 1559 the dot merely follows integers, except in the case of mixed numbers when it is sometimes omitted (f. 15, r.; 1559, f. 21, v.).

Illustrations of problems are found in the margin, as was common in the early texts. There are checks by 9's and 7's.

Four methods of multiplication are given, and the short method of dividing by 10, illustrated as follows: $35|7$ (f. 9, v.).

$$\begin{array}{r} 35\ 7 \\ \underline{10} \end{array}$$

It is singular that in the 'a danda' method of division, the divisor is placed above the dividend. He also puts dots

1. L. L. Jackson, The Educational Significance of Sixteenth Century Arithmetic, (New York, 1906), p. 28.
2. Dr. Siegmund Günther, Geschichte der Mathematik, v. 1 (1908), pp. 318, 319, chooses the work of Tagliente from among the early Italian arithmetics, and gives a detailed account of it.

above the numbers in the dividend as soon as carried down (f. 11, v.). Fractional lines are still broken, and guiding lines are used in all the fundamental operations with fractions.

His arithmetic is followed by a few folios relating to geometry.

6. Italian: Francesco Feliciano (No. 29.) (1550, 1563, 1669, 1692)

In 1526 appeared an Italian arithmetic (No. 29.) which had many subsequent editions, one even as late as 1692. This work, which was in use for over a century and a half after it was first published, is worthy of more than the one reference given by Cajori.¹

The text of Feliciano is replete with historical references, among them Aristotle, Euclid, Boethius, Sacrobosco, Borghi, and Pacioli. Their influence is shown in the first part of the book, which opens with a discussion of the various kinds of numbers and is, in general, commercial in character. The 'Scala grimaldelli', as it was called, might, in its day, be considered as a modern, condensed but complete, practical text; one which, like Tagliente's arithmetic, would have much influence on the later teaching of elementary mathematics.

We refer to two editions of each of the two centuries in which it was used. Like his contemporary, Cataneo, in the earlier edition of 1550 Feliciano encloses integers and fractions in the running text by dots. He is, however, inconsistent in placing a dot between the integer and fraction of a mixed number, omitting it, or using it, sometimes on the same page (f. 6, r.). In 1563, he merely follows numbers by a dot, occasionally by a comma.

Feliciano uses nulla or zero for 0 (f. 8, v.). The usual X in his proof by 9's and 7's is written as follows:

$$\begin{array}{r} 6|0 \\ 0|0 \end{array}$$
 (f. 11, r.).

Addition and subtraction are performed in the margin with denominate numbers only. Of the six methods of multiplication illustrated, one is an uncommon variation, that of writing the numbers of the multiplicand in reverse order, and then multiplying them from left to right by those of the

1. Cajori, Notations, p. 258, refers to his work but once in connection with the use of X in addition and subtraction of fractions.

multiplier from right to left; thus, to multiply 9876 by 6543 (f. 12, r.):

It is curious to note here that in 1669, the editor writes 9876 in the usual order (p. 23.).

6789
6543
29628
39504
49380
59256

The second part of this arithmetic is devoted mainly to algebra. There Feliciano writes B for square root, B cuba and B relata for cube and fifth roots respectively (f. 40, v.). The usual + and - are symbols for plus and minus (f. 46, r.) (in 1669, he uses men and più), as also co., ce., and cu. for cosa, censa, and cubus (f. 60, v.).

An addition of twenty-three problems of algebra is made to the second part of the text in the 1669 edition. It is there that we find plus indicated by the cross of the form +, minus by a long bar. A and AA are used to represent cosa and censo, and in the case of more variables, recourse is had to B, C, as far as D (1669, p. 185), following Vieta.

D. E. Smith in his Rara Arithmetica does not take note of these additional problems of algebra in this edition of 1669, nor does he describe the 1692 edition, which is essentially the same, except that the eight and the twelve-pointed Maltese cross are used interchangeably (1692, pp. 168, 169).

7. Italian: Giovanni Sfortunati (No. 85.) (1561)

The seven editions of Sfortunati's book Nuovo Lume. Libro di Arithmetica, prove that he was a popular writer. In his text he followed along the lines of the Italian writers of business arithmetics of his time.¹

Sfortunati in 1561 grouped the digits in numeration as we do today. Notable, too, is the fact that although, as Cajori states (pp. 254, 257, 260.), he used the form X in the process of false position, in proportions involving fractions, and in division of fractions, he does not use it in his check by 9's and 7's. It is to be noted also that dots follow only the integers in the running text.

In division, the divisor is placed above or below the dividend, and dots are placed after the numbers of the dividend when brought down (ff. 18, v., 20, v.). Division of
1. D. E. Smith, Rara, p. 174, also mentions the influence of Borgi, Feliciano, Pacioli, and Calandri upon the arithmetic of Sfortunati.

84789 by 10 is carried out thus: 8478|9 (f. 17, r.).

As does Feliciano, he adds a part on geometry, but there is no section on algebra.

8. English: Cuthbert Tunstall (No. 91.)
(1538, 1544)

The treatise, De arte supputandi libri quatuor, Cuthberti Tunstalli, written in Latin, first appeared in London in 1522, while the seven subsequent editions were published in Paris or Strasburg. It was the work of the Englishman Tunstall, a classicist, who included in his work much of the business arithmetic of Italian authors.

It is strange that a book so prolix that, among other things, it included addition, subtraction, and division tables (pp. 21, 29, 45.)¹ should have been in such great favor among the mercantile class. This may, perhaps, be partly explained by the fact that Tunstall was a prominent man in the Church and State, and that he was greatly influenced by Pacioli.²

Tunstall uses the term partitio (division) and subductio (subtraction) (p. 7.), and calls zero siphram (p. 9.). He mentions duplication (duplation) and dimidiation (mediation) (pp. 43, 59.), but does not treat them as separate species. He is most careful to arrange numbers directly under each other with the aid of a vertical dash (p. 55. In the 1544 ed. the dash is slanted.). For grouping digits in the extraction of roots he uses dots above.

Our method of subtraction is clearly displayed in the following manner (p. 30.):

$$\begin{array}{r}
 2 \ 9 \ 10 \ 10 \\
 \begin{array}{c} | \\ 3 \end{array} \begin{array}{c} | \\ 0 \end{array} \begin{array}{c} | \\ 1 \end{array} \begin{array}{c} | \\ 0 \end{array} \\
 \hline
 1 \ 1 \ 1 \ 1 \\
 \hline
 1 \ 8 \ 9 \ 9
 \end{array}$$

There are generally no broken fractional lines; occasionally the lines are omitted altogether after the first

1. Kästner, Geschichte der Mathematik, v. 1, pp. 94-96 refers to these tables.
2. Tunstall and Recorde are listed as the representative English writers of arithmetic in the sixteenth century by Günther, Geschichte der Mathematik, v. 1, pp. 343, 344, and also by Cantor, Vorlesungen über Geschichte der Mathematik, v. 2, pp. 475-480.

fraction (pp. 75, 80.). The 1544 Strasburg edition presents only this minor variation, that the fractional lines are broken throughout (1544, p. 133.).

9. Austrian: Georg Peurbach (No. 67.)
(1539)

The Latin work Elementa Arithmetices (Venice, 1539) of the mediaeval writer, Georg Peurbach, is included (ff. 32-67.) in a volume with the Elementa Geometriae by Johann Vogelin. It contains merely a summary of the fundamental operations with at most two or three illustrative problems. It is surprising that a book so theoretical in character should have had about seventeen editions, and yet we must remember that it was written by the able mathematician, Peurbach, teacher of Regiomontanus, to explain the Hindu-Arabic notation, and that it embodies many points of superiority over other early texts.¹

Only integers in the running text are followed by dots, as are also the terms of a progression (f. 39, v.). Fractions are commonly left in improper form, with broken fractional lines (f. 52, r.).

Mediation and duplation strangely both precede multiplication (ff. 35, v., 36, r.). In the extraction of roots, numbers are grouped by placing dots above (f. 41, r.).

While this work of Peurbach is noted because of the number of editions, it adds very little as regards the development of notations.

10. French: Oronce Finé (No. 33, 34.)
(1555, 1587, 1670)

The first work of the French mathematician, Orontius Finaeus, Orontii Finaei Delphinatis, Promathesis, was written in Latin in 1532. It was divided into four parts: arithmetic, geometry, cosmography, and horography. This study is based in detail upon the 1555 Latin edition (No. 33.), which contains only the first of these parts, comparing it with the two Italian editions of 1587 and 1670 (No. 34.).

1. Gerhardt, Geschichte der Mathematik in Deutschland, (München, 1877), pp. 9, 10, gives an idea of the importance of Peurbach's work.

The part on arithmetic deals with integers, common fractions, sexagesimal fractions, and proportion. Zero is called tziphra (f. 1, v.). Both subtraction and multiplication are performed by the complementary method.¹

It is curious that Finé places the quotient between the dividend and divisor; thus, to divide 4761 by 23 (f. 18, r.): $\frac{4761}{207}$ Sometimes (f. 9, r.) he writes down the multiples of the divisor before dividing.

$$\begin{array}{r} 4761 \\ 207 \\ \hline 2333 \\ 22 \end{array}$$

It is important to note that this author omits the fractional line after the first fraction when multiplying fractions by one another; thus, $\frac{3}{4}$ times $\frac{1}{2}$ is written $\frac{3}{4} \frac{1}{2}$

(f. 20, v.). Guiding lines are used in all operations with fractions.

Finé's interest in astronomy and geometry led him to include a treatment of sexagesimal fractions in his arithmetic.

Finé devotes many folios to proportions, and makes use of an inclined bar in writing them, somewhat as follows: $\frac{8}{4}$, $\frac{6}{3}$, meaning our $8:4::6:3$ (f. 59, r.). He uses vertical bars to group the digits in the extraction of roots, $5|30|84|16$ (f. 11.).

In the Italian editions of 1587 and 1670, which are essentially the same, a few deviations from the notations of the Latin editions may be cited. Tziphra becomes zero (1587, f. 2, r.), and all abstract integers in the running text are followed by dots, as was customary in the early printed Italian texts.

In subtraction he explains the complementary method simultaneously with the usual method (*Ibid.*, f. 5, v.). Proportions are now indicated in the following manner (*Ibid.*, f. 72, r.):

$$\begin{array}{ccc} 8 & 12.10 & 15 \\ A \text{ --- } B.C \text{ --- } D \end{array}$$

11. French: Pierre Savonne (No. 82.) (1571, 1619)

Pierre Savonne, called Talon, wrote a French commercial arithmetic in 1563, which, although not included among the sources of either Cajori or Tropfke, was so popular that an edition appeared as late as 1672. Our study is based upon

1. Tropfke, *Geschichte*, v. I, 1921, p. 29, mentions this method used in subtraction in the 1535 ed., p. 28. Cajori refers to his use of complementary multiplication in 1532 (p. 264.).

examination of the editions 1571 and 1619, both printed at Lyons, the commercial center of France.

Savonne devotes most of his arithmetic to banking and exchange, and so offers few marked peculiarities in notation. The commercial nature of the book accounts for his introduction of numerous short methods of multiplication and division.

In the 1571 edition he speaks of the form (0) 'qui s'appelle nulle, & entre marchans zero,' whereas in the 1619 edition Savonne merely calls it zero (f. 1, r.; 1619, p. 1.).

This text uses the Greek and the Latin cross respectively in the check by 9's in the two editions under comparison (*Ibid.*, f. 9, r.; p. 33.). In both, subtraction is performed by the complementary method and dots are placed to the right of the borrowed numbers, thus (f. 6, r.):

$$\begin{array}{r} 6 \cdot 4 \cdot 0 \cdot 6 \cdot 0 \\ 2 \ 7 \ 8 \ 7 \ 4 \\ \hline 3 \ 6 \ 1 \ 8 \ 6 \end{array}$$

In division the divisor is placed below the dividend, a survival of the scratch method of division.

It is strange that we find the use of broken fractional lines in the later edition (1619, p. 48.), and that the fractional lines are quite frequently omitted altogether in the performance of the fundamental operations with fractions (*Ibid.*, p. 54.).

12. French: Pierre de La Ramée (No. 74, 75.).
(1599, 1613)

Of the several books on mathematics written by Petrus Ramus, we shall consider his Arithmeticae Libri Duo, one with a commentary of Schonerus in 1599 (No. 74.), the other with Snell's commentary of 1613 (No. 75.), and refer only to the peculiarities in symbolism not noted by Cajori or Tropfke.

This work, unlike the arithmetic of Savonne, was too theoretical to meet the needs of the commercial world, and yet it was popular in the Latin schools for about half a century.¹

The edition by Schonerus (No. 74.) contains the arithmetic (two books) and geometry of Ramus, and it includes treatises by Schonerus on the Greek theory of numbers, and

1. P. Treutlein, "Das Rechnen im 16. Jahrhundert", Abhandlungen zur Geschichte der Mathematik, v. 1, p. 17, in describing this arithmetic, says: "in welcher im Ganzen nach der uns heute geläufigen Reihenfolge die Hauptsache des Zahlenrechnens gut dargestellt ist."

on sexagesimal fractions (as in Finé). The 1613 edition (No. 75.) includes Snell's commentary on Ramus' two books of arithmetic with his addition De re nummeria (1635).

In Ramus' arithmetic (Schonerus), the digits in numeration are grouped by dots (1599, p. 2.); in Snell's, by commas (1613, p. 2.). Zero is referred to as circulus in both (1599, p. 1; 1613, p. 2.). His plus sign is a Greek cross with the horizontal stroke much longer than the vertical one, thus: + (1599, p. 182.).

Subtraction is performed from left to right, and the remainder is placed above after scratching out the numbers, thus (Ibid., p. 6.):

$$\begin{array}{r} 8 \ 7 \\ \text{A} \ \text{B} \ \text{Z} \\ \text{B} \ \text{A} \ \text{B} \end{array}$$

Again the divisor is placed below the dividend, and a lunar sign on the right for the quotient, somewhat in this arrangement (Ibid., p. 20.):

$$\begin{array}{r} \chi \\ \text{Z} \ \text{B} (2 \ 6 \\ \text{B} \ \text{B} \end{array}$$

Division of 7200 by 100 is performed in this manner (Ibid., p. 25.):

$$\begin{array}{r|l} \text{Z} \ \text{Z} & 0 \ 0 \\ \text{Z} \ \text{Z} & 0 \ 0 \end{array} (72$$

When taking a fraction of a fraction, the fractional line is omitted after the first fraction (Ibid., p. 39.). Ramus uses a dot to group the digits in the extraction of roots (Ibid., p. 154.).

13. Italian: Georgio Lapazzaia (No. 56.) (1601)

Again it is desirable to draw attention to a work which was in use for over two centuries. The arithmetic by the abbot Georgio Lapazzaia was first edited in Latin in 1566, and Riccardi records an edition as late as 1784, thus extending over three centuries. This longevity is probably due to the popular style of the work, its commercial nature, and the influence of the Church schools of that time.

The Libro d'Arithmetica e Geometria, the particular edition under consideration, published at Naples in 1601,

contains very little that is singular in the way of notation. The grouping of digits in numeration is indicated thus (P. 7.)

$$\begin{array}{ccccccccccccccc} 3 & 7 & 8 & 5 & 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \\ \vdots & & & \vdots & & & \vdots & & & & & & \end{array}$$

Lapazzaia is consistent in using dots below when grouping numbers for root extraction (p. 117.).

There are found also the familiar two intersecting lines for results in casting out the 9's and 7's, and the guiding lines in carrying out the fundamental operations with fractions.

Fundamental operations are carried out first with abstract, then with denominate numbers. Subtraction by the complementary method and division by the galley method offer nothing novel in representation.

The last part of the work deals with mensuration.

14. Italian: Guiseppe Figatelli (No. 32.) (1678, 1699)

Another 'centenarian' which escaped the attention of Cajori is that of the Italian Figatelli, particularly noteworthy because it is rich in notation.

Figatelli wrote his first book on arithmetic in 1664, and Riccardi gives these subsequent editions: 1680, 1686, 1692, 1699, a sixth impression without a date, 1737, 1738, 1778, 1791 and 1797. He does not mention the second impression, Venice, 1678, upon which our study is based.

This arithmetic, to which is added algebra, was written not only for merchants, but also for teachers. The Italian symbolism with its $\tilde{p}$ and $\tilde{m}$, his use of nu., co., qu., cu., for powers, and the crossed capital R as his basic symbol for roots, his various methods of multiplication (per colonna, baricocolo, ripiego, crosetta, quadrilatero, and piramide) (pp. 6-10.) all show the persistence for centuries of the familiar notations of Pacioli.

Figatelli, in both the 1678 and 1699 editions, includes Roman notation, indicating the thousand-fold value of a number by placing a horizontal line above it, thus: $\bar{V} = 5000$. For the sake of abbreviation he writes 400 as $\overset{c}{1111}$, 6,000

as ^Mvi, placing the C and M above the number (pp. 3,4.).

Multiplication 'per crosetta' with the check by nines is displayed as follows (p. 8.):

$$\begin{array}{r} 5 \quad | \quad 2 \\ | \quad \times \quad | \\ 4 \quad | \quad 6 \\ \hline 2 \quad 3 \quad 9 \quad 2 \\ \cdot \cdot \cdot \end{array}$$

Notice that the dots below are the numbers to be carried.

Figatelli's further liberal use of dots for various purposes of convenience is best shown in his illustration of division of denominate numbers, thus (p. 16.):

$$\begin{array}{r|l} \text{Denari} & 2 \ 4 \ 0 \ 0 \ 0 \ 0 \\ 9 \ 6 \ 0 & \underline{1 \ 9 \ 2 \ 0 \ . \ .} \\ & . \ 4 \ 8 \ 0 \ 0 \\ & \underline{4 \ 8 \ 0 \ 0} \\ & . \ . \ . \ . \ 0 \end{array} \quad \begin{array}{l} 2 \ 5 \ 0 \text{ Scudi} \end{array}$$

In the extraction of roots, dots above are used for grouping; he uses them also in the writing of large numbers, thus (p. 4.):

$$\begin{array}{cccccccccccccccc} & 3 & & & 2 & & & & 1 & & & & & & & & \\ 2 & 6.4 & 2 & 3.5 & 7 & 0.6 & 4 & 0.3 & 2 & 5.4 & 8 & 9.2 & 2 & 5 \end{array}$$

Figatelli introduces the sign * for indicating commensurable (communicanti) radicals (p. 183.); thus there occurs this problem in addition of radicals (p. 184.):

$$\begin{array}{rcl} \text{A s\`omare co } R_x & 20 * \tilde{p} & 3 \\ \text{questa } R_x & 5 * & \\ \text{Far\`a } R_x & 45 & \tilde{p} \ 3 \end{array}$$

The notation for proportion is very clumsy; for example (p. 200.):

$$\begin{array}{c} 24 \\ \overbrace{12 \grave{a} 4 \text{ ~~12~~ 6 \grave{a} 2} \\ 24 \end{array}$$

15. German: Simon Jacob (No. 54.)
(1612)

Jacob's two arithmetical works must have met with favor between 1550 and 1600, for there were nine editions during that time. These were arithmetics of the Adam Riese type, commercial in nature, extensive in applied problems,

and still devoting a few pages to reckoning 'auff den Linien.' Jacob was the most important German reckoning master of the last half of the sixteenth century.¹

The particular edition of Ein New und Wolgegründt Rechenbuch used was printed at Frankfurt in 1612. The preface to the reader is dated 1552, and the colophon gives the date 1600. (The first edition listed by Smith is of 1560).

There are several things of note as regards symbolism. All integers of the running text are followed by dots. Zero is called nulla (f. 1, r.). The digits of large numbers are grouped in this manner (f. 1, v.): 4/512̇/789̇/087. Jacob makes use of dots above as a separatrix for the extraction of roots. There are explanations of proba by 9's, 3's, 7's, 8's, 11's, etc.

In connection with the reduction of fractions to lowest terms, the author explains divisibility by numbers below 10 (ff. 20, 21.). Tropske² refers to Jacob's test of divisibility by 7 as that used by Ibn Albannā.

Jacob uses the words of Rudolff for powers of the unknown, namely, rad., zens, cub, zensdezens, etc.; for roots, radix quadrata, radix cubica, radix zensdezensico, but he has no symbols for them (f. 89 ff).

Outside of his use of the plus sign in the form of a Maltese cross in the rule of false position, Jacob uses it only in writing the powers of a binomial. He writes the result of $(x+1)^2$ thus (f. 106, r.): 1. zens ✠ 2. rad. ✠ 1.

16. German: Johann Hemeling (No. 47, 48.)
(1667, 1678)

The Historische Rechne-kunst (No. 47.) of Hemeling contains one hundred historical problems; interspersed we find couplets that are religious in character.

Hemeling uses one, two, three, or more dots superimposed to mark the groups of digits in numeration, thus (p. 4.) 10000000, and also to indicate borrowing in subtraction (p. 8.):

$$\begin{array}{r} 8 \ 2 \ 4 \ 8 \ 0 \ 0 \\ 4.3.5 \ 4 \ 0 \ 0 \\ \hline 3 \ 8 \ 9 \ 4 \ 0 \ 0 \end{array}$$

1. L. L. Jackson, The Educational Significance of Sixteenth-Century Arithmetic, (New York, 1906), p. 28.

2. Tropske, Geschichte, v. I, 1921, p. 106.

It is interesting that he should write the partial products of 1 2 3 4 5 by 1 4 3 2 thus (p. 11.):

$$\begin{array}{r}
 1\ 2\ 3\ 4\ 5 \\
 4\ 9\ 3\ 8\ 0 \\
 3\ 7\ 0\ 3\ 5 \\
 \hline
 2\ 4\ 6\ 9\ 0 \\
 \hline
 1\ 7\ 6\ 7\ 8\ 0\ 4\ 0
 \end{array}$$

This order has been proposed very often as more useful in lending itself to cutting off non-significant digits.

We find Hemeling using the ordinary plus sign, and $\div$ for minus (p. 177.). 1 R is his symbol for the unknown X (p. 174.).

Hemeling wrote various other works, but our study is based upon the Selbstlehrende Rechenschul or Selbstlehrendes Rechnebuch of 1678, of which there is a revised edition by E. A. Talgener as late as 1817, an indication of the probability that it was still used in many schools.

In this work the dots used for grouping in numeration are slightly changed in position, thus (1678, p. 21.): $\dot{2}\ 4,5\ 2\ \dot{1},5\ 9\ \dot{4},4\ 9\ 7$. The otherwise impossible subtraction is performed by the addition of one to the next highest number in the subtrahend, but now Hemeling indicates this addition by placing the dot below the numbers.

There are addition and multiplication tables, and the additions are performed downwards. Hemeling checks by the inverse operations, and does not recommend the check by nines.

Again Hemeling uses $\div$ for minus,¹ and we still find this redundant symbol of subtraction in the nineteenth-century revised edition. Occasionally we find him using the Maltese cross for plus (pp. 381, 478, 483.), but the usual sign is +.

Hemeling deals extensively with denominate numbers, and his book is largely commercial in character.

17. English: Jonas Moore (No. 63.)
(1660, 1681)

Moore's Arithmetick, as it is called, had three editions. All of them contain vulgar arithmetic, including the use of Napier's rods, logarithms, and decimals, besides

1. See Cajori, Notations, p. 242.

algebra. Moore also wrote several other mathematical works, but our discussion is largely confined to Moor's Arithmetick in Two Books (London, 1660), which abounds in notation much like the works of Oughtred, his contemporary. Besides the seven references made by Cajori to Moore's 1660 and 1688 editions, many other symbols are worthy of note.

Moore treats the fundamental operations in integers, decimals, and denominate numbers simultaneously. After proportion, fractions, and algebra, follow two treatises, the Ellipsis and Conical Sections, and finally a table of the squares and cubes of numbers under 100, the fourth powers of numbers under 300, and the fifth and sixth powers under 200. Zero is called cypher (p. 2.), and large numbers are divided into groups of three by a colon, thus: 1:758:327:821 (p. 4.). Moore briefly speaks of casting out 9's as a check for addition.

Various ways of writing the decimal fraction 245.1234 are given by Moore (pp. 10, 11.): 245.1.2.3.4., 245,1234, and 245,123; but the 'best and most distinct way' is by means of a 'separatrix, a separating line', thus: 245|1234. In some of the work following, however, Moore uses an elevated and inverted comma instead (p. 20.), which is the only decimal symbol which Cajori attributes to him. He gives the sign of addition as St. George's cross, elevated and small (p. 16.), thus: $+$. The sign of multiplication is referred to as St. Andrew's cross, but it is sometimes made much like the plus sign, whereas throughout most of his work Moore correctly uses the small elevated $\times$. Cajori (p. 197.) refers to this edition (p. 108.) for his use of the letter x for multiplication, but Moore uses no symbol whatever for multiplication on the indicated page.

The symbol for subtraction is first represented as a very small horizontal line, sometimes much longer, and at other times as two short successive dashes (pp. 16, 84, 85.). Moore also uses $\pm$ for plus and minus (p. 191.).

Bypartition (mediation) is demonstrated (p. 26.) from left to right, but the author states (p. 27.) that 'it will be of most speed to do it from the right hand.' There are illustrations of duplication, triplication, and reduplication (pp. 26, 27, 28.).

Moore relates that he taught his pupils to remember multiplication by 5, 6, 7, 8, and 9, as follows (p. 32.):

let either hand closed signifie 5, by raising the thumb it will be 6, by raising the thumb and one finger 7,...; which done, set the two figures, whose

Product you desire, on both the hands; all those figures which are up are to be accounted so many tennes, and multiply those which are down by themselves, adding it to the tennes, it gives the desire.

This is a form of complementary multiplication still practiced by the Wallachians at the end of the nineteenth century.¹

In this work dots are placed below the numbers in the dividend as soon as they are carried down, but dots above are used to mark the periods for square root (pp. 48, 65.). This edition (p. 2 of the 2nd. part) also uses $\div$ for the continual proportion.

The powers of the root are represented in various ways (p. 7.), by means of the great letters, the lesser characters, and by exponents, according to the way of Oughtred, of Harriot, and of Descartes, respectively; for example, the third power of A is written $\underline{Ac}$, $\underline{aaa}$, and $\underline{a^3}$. The square root of any quantity is expressed by $\sqrt{\quad}$, cube root by $\sqrt[3]{\quad}$, fourth root by $\sqrt[4]{\quad}$, etc., according to Oughtred (p. 11.).

Moore gives a list (p. 12.) of the letters appointed by him as symbols for proposed quantities, as follows:

As if A and E signifie two numbers: let Z signifie the summe of these numbers, X the difference, $\mathcal{A}$ the rectangle or multiplication, $\frac{P}{Q}$ the proportion, K the summe of the squares, L the difference of the squares, M the summe of the cubes, and N the difference of the cubes of them.

Later (p. 72.) he gives almost similar symbols in the lesser characters.

A few of the more important variations in another work of Moore (1681) are well worth noting. In his A New System of the Mathematicks, in which is included 'Arithmetick, as well Natural and Decimal, as in Species, or the Principles of Algebra', Moore arranges the nine digits in decreasing order. In this work the dot displaces the colon in grouping the digits in numeration. More noteworthy is the fact that in this later work Moore consistently uses the dot as a decimal symbol. The comma is used only in the case of

1. P. Treutlein, "Das Rechnen im 16. Jahrhundert", Abhandlungen zur Geschichte der Mathematik, v. 1, p. 48, refers to an article by Dr. Pick (1874) in which similar instructions are given for multiplication. This process may also be found in the treatises of the thirteenth century.

logarithms. These additional signs are listed: $\div$ (division), $\odot$ (involution), $\ominus$ or ω (evolution), all used in Brancker's translation of Rahn (1668).

18. Swiss: John Alexander (No. 1)
(1693)

The Synopsis Algebraica, a posthumous work in Latin by John Alexander, appeared at London in 1693; it was translated into English by Sam. Cobb sixteen years later. Like the works of Vieta and Oughtred, to whom he refers (pp. 3, 23.), this book is important to us here because of its notations.

Alexander uses + and -, but not $\times$ or $\div$. He gives = or Descartes' sign $\propto$ as symbols for equality (p. 2.), but uses the former throughout his work. Cajori¹ says that he used $\therefore$ for therefore, whereas we find it was $\therefore$. Probably an omission in the printing (p. 7.) accounts for this error.

This author writes the proportion $a.b::b.x$ or $a=b:b=x$ (p. 16.), although Cajori states that he expressed geometrical ratio by a dot, $a.b$, and also by $a=b$.² In other places, however, Cajori refers to the correct symbol = (Ibid., pp. 287, 289.).

Whereas others designate involution and evolution with the Archimedean spiral and the double epsilon ($\odot$ and ω), Alexander says he uses exponents (p. 19.). However, square, cube, and fourth roots are written $\sqrt{q.}$, $\sqrt{c.}$, and $\sqrt{qq.}$ respectively (pp. 24, 28, 32.). To express the square root of a quantity he makes use of the vinculum with the radical sign, in this manner (p. 35.): $\sqrt{a^2 + b^2}$.

B. Topical Survey of the Use of Notations

1. NOTATION AND NUMERATION

1. Finger and counter reckoning. The finger symbolism of number is found in the works of Baeda, 'The Venerable Bede' (c. 673-735), was practiced in both the East and West during the Middle Ages, and is mentioned by several sixteenth-century arithmeticians. The 1491 Calandri (No. 14.) was probably the first printed work to illustrate finger symbolism (not the 1494 Pacioli as given in Smith's Rara,

1. Cajori, Notations, v. II, p. 282.

2. Cajori, Notations, v. I, p. 276.

p. 56.). Aventinus wrote a most complete treatise (Abacus atque vetustissima veterum latinorum per digitos..., Regensburg, 1522.) on numerical finger symbolism using both fingers and arms. (See P. Treutlein, "Das Rechnen im 16. Jahrhundert", Abhandlungen zur Geschichte der Mathematik, v. 1, p. 22, for a discussion of this work.)

The treatment of finger symbols by Tagliente (1520) has been noted above. Another writer who discusses this subject is Jan Bronkhorst of Holland, called Noviomagus from his birthplace. In his De numeris libri II (No. 65.) the first book is 'de Logistice' or 'de arte computandi.' There is an illustration of finger reckoning (f. 19, v.), practically the same as that described by Pacioli, with the exception of the symbols for 100, 200, and 1000.

As to counter or abacus reckoning, not only Husswirt (1501) (mentioned above) but also Reisch (No. 77.), Blasius (No. 8.), and Feme (No. 31.) give early treatments. The titles such as Rechenbiechlin auf den linien by Köbel (1514), and Rechnung auff der Linien und Federn by Riese (1522) indicate the work on the lines. It was not uncommon to find the title-page bearing a woodcut showing computing with counters, as in Köbel's Rechenbiechlin (1514), and Recorde's Ground of Artes (1558). Noviomagus (No. 65, ff. 32, 36.) also briefly performs all fundamental operations upon the line abacus. The German Ulricus Regius (No. 76, ff. 94-100.) closes his book with several folios on the use of counters; Johann Scheubel (No. 83, pp. 13, 14, 25, 31, 43-45, 54, 55.) speaks of numeration and of the fundamental operations 'per calculos.' In the early seventeenth century Sesen (No. 85, p. 5.) gives this topic extensive treatment.

From such works we see that line reckoning was common in the first half of the sixteenth century, especially in Germany and Holland. One of its last serious advocates is Lucas à S. Edmundo, who as late as 1697, devotes a chapter (No. 58, pp. 214-220) to it, entitled 'Arithmetica Calculatoria', with illustrations of counter reckoning. A detailed account of the method of casting with counters (jettons) as found in Legendre's work of 1753 (No. 57.) is given by F. P. Barnard (The Casting-Counter and Counting Board, p. 313.). It is interesting to note that counter-reckoning is not included in the earlier (1690) edition of Legendre's arithmetic.

2. Roman Numeral symbols. Blondel (No. 9, Tome Second, pp. 56-59.), in his chapter on numeration of the Romans,

speaks of the probable origin of the characters of the Roman notation, X, C, M, V, L, and D. X, he says, was formed by crossing the line I when, in counting, people came to the end of their fingers. V is given as the upper part of a bisection of X; and L as the lower half of 100 made in this manner $\square$ (seen on ancient inscriptions); and D was the right half of the symbol for 1000, $C|D$.

There is frequent use of the principle of subtraction in the case of IX, rarely of IV. We find the use of IX and IIII in Regius (No. 76, f. 49, v.) and Steinmetz (No. 86, f. 11, v.). The same notation appears in Noviomagus (No. 65, f. 20, r.), Vuelpius (No. 98, f. 6, r.), and as late as 1695 in Behme (No. 7, p. 3.). The latter writer gives, however, 34 as XXXIV. The use of this principle is extended to 8 as IIX and 80 as XXC by Waser (No. 99, f. 9.) and Lucas (No. 58, f. 8, r.) respectively, to 400 as CD, and to 900 as C ∞ by Waser (No. 99, f. 9.). Blondel (No. 9, *ibid.*) uses both IIII and IV, VIII and IIX, VIIII and IX, XXXX and XL, and the forms LXXXX and XC. He also mentions examples in ancient inscriptions where the subtractive principle is used in writing 8 as IIX and 18 as XIIX, but states that this is very rare.

The principle of multiplication, according to which IIM stands for 2,000, occurs in Noviomagus (No. 65, f. 20, v.), Vuelpius (No. 98, f. 7.), Steinmetz (No. 86, f. 12.), Lucas (No. 58, f. 8, r.) and in Blondel (No. 9, *ibid.*). The indication of the thousand-fold value of a number by a dash placed above has been noted previously in the case of Figatelli. Monier (No. 62, pp. 3.) gives these two representations for 10,000, XM and $\bar{X}$. Figatelli writes $\overset{M}{M}$ for 6,000, and Behme (No. 7, p. 3.) I^m for 1,000. VI

There are various symbols for 1,000 due to the fact that for this the printer had to make up a separate character. Lucas (No. 58, f. 8, r.) and Blondel (No. 9, *ibid.*) use M or $C|D$; to these Monier (No. 62, p. 3.) adds $\bar{I}$. Behme (No. 7, p. 3.) gives I^m and $C|D$, and Regius (No. 76, f. 49, v.) $\bar{M}$. Unicornio (No. 93, f. 4, v.) writes M, $C|D$, or ∞ , and Waser (No. 99, f. 9.) gives these three symbols, $C|D$, ∞ , and CX ; in fact we find the symbol ∞ used in writing the date 1603 as on the title-page of the latter's book. The principle of multiplication is used by Monier (No. 62, p. 4.) in writing 1,000,000 as MM, whereas Lucas (No. 58, f. 8, r.) uses the dash, $\bar{M}$. Waser (No. 99, f. 9.) and Behme (No. 7, p. 3.) give the old Roman symbol CCCCI ∞ for one million.

There are several instances of the prolongation of the last I in the units for the purpose of preventing the subsequent additions of a further unit, as $\text{iii}|$ for 4, and $\text{ii}|$ for 2, in Vuelpius (No. 98, f. 6, v.) and Blondel (No. 9, *ibid.*) respectively. Feme (No. 31.) and Sesen (No. 84, p. 6.) write $\text{iii}|$ for 4, using printed i's. (See account of the Treviso Arithmetic (1478) in D. E. Smith's Source Book in Mathematics (New York, 1929), p. 3, wherein i's are also used for ones.)

3. Hindu-Arabic numerals. In grouping digits in numeration, vertical bars, commas, dots, numbers, strokes, arcs, and combinations of these occur in the arithmetics under examination.

Cajori says (p. 58.) that Gemma Frisius wrote 24 456 345 678 (without the use of any separatrix) in his 1540 edition; it is noteworthy that he uses the vertical bar as a separatrix of this same number in his 1549 and 1550 editions, as has been mentioned previously. Sfortunati (1561) and Irson (No. 53, p. 11.) mark a number as we do today, while Forestani (No. 35.) and Hodder (No. 51.) employ the dot in place of the comma. Isaac Greenwood's book of 1729, the earliest native American arithmetic, used both of these latter forms (See D. E. Smith, History of Mathematics, v. 2, p. 87.).

Dots below as a sign of separation are used by Lapazzaia, as illustrated before. Tartaglia (No. 90, f. 7, r.) in his General trattato di numeri of 1556 writes dots above, thus:

$\overset{\cdot\cdot\cdot}{2}\overset{\cdot\cdot\cdot}{3}\overset{\cdot\cdot\cdot}{4}\overset{\cdot\cdot\cdot}{5}\overset{\cdot\cdot\cdot}{6}\overset{\cdot\cdot\cdot}{0}\overset{\cdot\cdot\cdot}{0}\overset{\cdot\cdot\cdot}{7}\overset{\cdot\cdot\cdot}{8}\overset{\cdot\cdot\cdot}{4}\overset{\cdot\cdot\cdot}{0}\overset{\cdot\cdot\cdot}{0}\overset{\cdot\cdot\cdot}{0}\overset{\cdot\cdot\cdot}{0}\overset{\cdot\cdot\cdot}{3}\overset{\cdot\cdot\cdot}{0}\overset{\cdot\cdot\cdot}{5}\overset{\cdot\cdot\cdot}{3}\overset{\cdot\cdot\cdot}{2}\overset{\cdot\cdot\cdot}{1}$. Steinmetz (No. 86, f. 10, r.) marks every third digit with a dot, thus $7\dot{8}96\dot{5}432$, as also do Graffenried (No. 42.) and Sesen (No. 84.). We find it interesting to note the various markings in Clavius (No. 24, pp. 10-12; No. 25, 1586, pp. 13, 14; No. 25, 1618, pp. 2-4; No. 25, 1686, pp. 10, 11.). In the 1585, 1586, and 1618 editions he employs dots and numbers in several ways, thus:

$$4\overset{2}{2}3\overset{2}{2}9\overset{0}{0}8\overset{9}{9}56\overset{2}{2}8\overset{0}{0}$$

$$4\overset{4}{2}3\overset{2}{2}9\overset{0}{0}8\overset{9}{9}56\overset{2}{2}8\overset{0}{0}$$

$$4\overset{2}{2}3\overset{2}{2}9\overset{0}{0}8\overset{9}{9}56\overset{2}{2}8\overset{0}{0}$$

In 1686 he uses only numbers, as in $4\overset{2}{2}3\overset{2}{2}9\overset{0}{0}8\overset{9}{9}56\overset{2}{2}8\overset{0}{0}$ and $4\overset{2}{2}3\overset{2}{2}9\overset{0}{0}8\overset{9}{9}56\overset{2}{2}8\overset{0}{0}$. Cajori (p. 59.) mentions only one method of marking a large number in the 1583 edition, namely $4\overset{4}{2}3\overset{2}{2}9\overset{0}{0}8\overset{9}{9}56\overset{2}{2}8\overset{0}{0}$, whereas the 1586 edition, which is a

translation of that of 1583, gives the three symbols for grouping mentioned above. Hemeling, similarly, uses dots above in his work of 1667, but dots and commas in 1678. Lucas (No. 58, p. 2.) uses dots above and below with strokes, as in $482\dot{3}1\dot{5}6789\dot{2}43$. Figatelli, as noted before, has dots and numbers, while Behme (No. 7, p. 3.) separates the digits in this particularly peculiar and profuse manner:

$$17\dot{6}59\dot{7}67\dot{4}89\dot{4}346$$

Vertical bars and dots occur most frequently as symbols for separating numbers into periods of two digits in the process of extracting square root. Besides Finé, whom we have mentioned earlier, Blondel (No. 9, p. 123.) uses the vertical bar as separatrix somewhat in this manner:

$$57|86|42($$

There are several adherents of dots below the digits, as Gemma, already cited, Aurel (No. 3, f. 41, v.), Clavius (No. 24, p. 309; No. 25, 1586, p. 274; No. 25, 1618, p. 268; No. 25, 1586, p. 306.) in all his editions, Peverone (No. 68, p. 55.), and Alsted (No. 2, p. 73.). For the use of dots above we have referred to Peurbach (1539), Jacob (1612), and Figatelli (1678). Regius (No. 76, f. 77.), Scheubel (No. 83, f. 77.), Steinmetz (No. 86, f. 58.), Bombolli (No. 11, p. 31.), Unicornio (No. 93, p. 86.), Napier (No. 64, p. 67.), and Harriot (No. 45, p. 130.) also use them. Harriot makes his dots very large and heavy. We find that Micyllus (No. 60, p. 69.) uses dots above in the extraction of cube roots, but for square roots he does not mark his numbers into periods in any way.

In many of the arithmetics of these two centuries we still find the integers of the running text enclosed by dots, as in Cataneo (1546), or merely followed by dots as in Peurbach (1539), Cataneo (1559), and Jacob (1612), as noted before. Tartaglia (1556) also places dots after integers, Tagliente (1520) after both integers and fractions, while Noviomagus (1544) uses them only after the integers in mixed numbers. Clavius again shows his fondness for dots, for he uses them after both integers and fractions.

2. SIGNS FOR ADDITION AND SUBTRACTION

4. Early Symbols. The Italian symbolism with its $\tilde{p}$ or $\underline{p}$. for plus and $\tilde{m}$ or $\underline{m}$. for minus, was used extensively by sixteenth-century and early seventeenth-century mathematicians in Italy. Unicornio (No. 93, p. 118.) uses $\underline{p}$. and $\underline{m}$. in

in his 1598 arithmetic, and as late as 1671 we find these same symbols in a work of the Spanish Puig (No. 72, p. 374). Cataldi (No. 17, p. 8; No. 18, p. 50.), of whom Riccardi lists at least thirty works, uses più and meno, sometimes $\tilde{p}$ and $\tilde{m}$, according to the crowded condition of the particular line. Feliciano's editions testify to the waning strength of the Italian symbols in the latter part of the seventeenth century, for $\tilde{p}$ and $\tilde{m}$ are used in his 1550 edition, but the Maltese cross in 1669, and the eight and twelve-pointed Maltese cross interchangeably in 1692, as we have seen earlier. However, Figatelli's works still adhere to $\tilde{p}$ and $\tilde{m}$ as late as 1678. (A lengthy account of the early history of the signs + and - is given by J. W. L. Glaisher in Messenger of Mathematics, v. 51, 1921, pp. 1-149. In it Glaisher credits Widman (1489) with the introduction of these signs; and shows that they had an arithmetical origin, not a commercial origin, as De Morgan (Camb. Phil. Trans., v. 11, 1871, pp. 203-212) believed).

5. Use of plus and minus in the rule of false. Several writers, in solving problems by false position, prefix plus or minus in various shapes to the error, as do Vuelpius (No. 98. On the last few folios we find + and - used to denote excess and deficiency in the error.), Lambeck (No. 55, p. 286. Lambeck also uses plus and minus in the solution of algebraic problems in the appendix, p. 300.), Jacob (1612. We have found the use of his plus sign (Maltese cross) only once outside of its application in the rule of false.), and Blondel (No. 9, p. 211, uses + and -). Gemma Frisius (1549, f. 36, r.) uses + and - in combination with guiding lines in his application of the rule of false position, somewhat in this manner $\times$; Cajori (p. 254.) only mentions the use of the guiding lines in the 1569 edition of Gemma Frisius.

6. Shapes of the plus sign. The Greek cross was usually used by Descartes (No. 28, p. 15.), as mentioned by Cajori (p. 237.). However, it is interesting to add that he also uses $\mathbf{+}$, the Greek cross with each of its four arms terminating in a line drawn across it, but he uses this form only in the terms of a fraction. This is probably due to the fact that the form + was too large to use in this case. In the same way the Spanish writer Ventura de Avila (No. 4, pp. 41, 69, 85.), uses a symbol other than the usual Greek cross when necessary in the numerator and denominator of a fraction, namely $\rightarrow$ and $\leftarrow$.

The Latin cross placed horizontally, $\rightarrow$, was used by Vieta (No. 95, p. 107; No. 96, p. 154.) in 1591, according to Cajori (p. 237.). This is the case in the 1630 algebra, but throughout his work of 1646, the writer finds that he uses $\rightarrow$ and $+$ interchangeably. Prestet (No. 70, p. 2.) also gives $\rightarrow$ as the symbol for plus, and we have said that Blondel used it in the rule of false.

The Maltese cross in its simple form, $\oplus$, we have seen, may be found occasionally in Hemeling. Lucas (No. 58, pp. 165, 286.) uses a form of it as an algebraic sign as well as in his rule of false. Feliciano (1692), we have seen previously, used an eight and twelve-pointed Maltese cross. The latter was also employed by Ceva in his work (No. 20, p. 9.).

A very fanciful twelve-pointed cross was used by Renaldini (No. 78, p. 80.). Cajori states (p. 238.) that he uses this form on page 80 and throughout the volume. We find three forms of the plus sign used by him on this same page, $\uparrow$, $\oplus$, $\star$, and throughout the volume these symbols appear with about equal frequency.

In a recent article (*Isis*, v. 17, Jan. 1932, p. 259.) by J. Pelsener, note is made of the successive use of three different signs to indicate addition, $\times$, $\bowtie$, and $\rightarrow$, in the treatise, *De Numeris Geometricis* of Schonerus, translated into English by Thom. Bedwell, London, 1614. The author of the article cites this as a unique case of the use of three different notations for addition in the course of a single work. The example of Renaldini given above is another such instance. The use of different symbols for the same operation is not uncommon.

7. Varieties of minus signs. The sign of subtraction used by Vuelpius (1544), Feliciano (1669), and Blondel (1699) was somewhat longer than the ordinary symbol $-$. A slight modification of it, — , was employed by Lucas (No. 58, p. 286.) and Renaldini (No. 78, p. 21.). Vieta makes this same symbol very heavy so that it renders the minus signs on a page very conspicuous (No. 95, p. 107.).

A short dash, sometimes longer, and again, two successive dashes were Moore's varieties, as noted above. Renaldini (No. 78, p. 40.) occasionally uses the latter form also.

Lambeck (No. 55, p. 300.) and Hemeling (1667) are two adherents of our present symbol for division as their symbol for subtraction, $\div$. Cajori (p. 242.) says the latter wrote $\div$ for minus, but in the example cited he uses $\div$,

probably a typographical error. The form $\div$ is, however, used by Curtius (No. 27.) as an operational symbol, whereas Graffenried (No. 42, pp. 701 ff.) uses $\div$ also with a qualitative meaning.

Thus we have found that in the early books the forms of type for plus and minus were not standardized. In many cases the symbol was adjusted in size to the space to be filled, and very frequently various forms were displayed on the same page.

3. SIGNS OF MULTIPLICATION

8. Early Uses of the St. Andrew's cross, but not as a symbol of multiplication of two numbers.

The process of two false positions. In the early use of St. Andrew's cross $\times$ in this process, each bar connects the two numbers which are multiplied, whereas later the two bars are considered as one symbol.

Reference has been made to the use of these guiding lines by Gemma Frisius in solving problems by double false position. We find similar settings of numbers in Micyllus

(No. 60, p. 129.), who writes $\begin{array}{c} 4 \times 1 \\ 2 \times 1 \end{array}$. Clavius, in his 1585

and 1586 editions, uses $\times$ for this purpose, but a large capital letter X in the later editions of 1618 and 1686. A slightly different scheme is found in Bassi (No. 5, p. 176.) who uses $\times$. Blondel (No. 9, p. 213.) uses the same scheme as Gemma and Micyllus.

9. Proportions involving fractions. There were various schemes used in the early arithmetics to serve as aids to computation. In the solution of proportions involving fractions we find the use of $\times$, single bars, horizontal lines, arcs, braces, and other devices. Bassi (No. 5, p. 76.) solves the proportion $\frac{2}{3} : \frac{3}{4} :: \frac{1}{2} : x$ thus:

$$\begin{array}{r} \frac{2}{3} \text{ — } \frac{3}{4} \text{ — } \frac{1}{2} \\ \frac{3}{8} \text{ — } \frac{3}{2} \\ \hline \frac{9}{16} \end{array}$$

10. Addition and subtraction of fractions. The cross $\times$ is used by Regius (No. 76, f. 85, v.) in reduction of fractions to equivalent fractions, thus:

$$\begin{array}{r} 8 \ 9 \\ \frac{2}{3} \times \frac{3}{4} \\ \hline 12 \end{array}$$

Bassi (No. 5, pp. 72, 73.) and Manelli (No. 59, pp. 25, 28.) use the large capital $\times$ on its side, $\times$, in the addition and subtraction of fractions. Zuchetta (No. 101, pp. 2, 9.) uses dotted lines for this same purpose in the following suggestive manner:

$$\begin{array}{r} \frac{2}{7} \quad \frac{3}{8} \\ \hline 16 \\ 21 \\ \hline 37 \\ \hline 56 \end{array}$$

11. Multiplication and division of fractions. Zuchetta (No. 101, pp. 10, 13.) uses his dotted lines in a less suggestive way in the multiplication of $\frac{2}{3}$ by $\frac{4}{5}$ as follows: $\frac{2}{3} \frac{4}{5} \dots \frac{8}{15}$; the division of $\frac{20}{3}$ by $\frac{3}{4}$ is represented better thus: $\frac{20}{3} \dots \frac{4}{3} \dots \frac{80}{9}$. Bassi (No. 5, pp. 74, 75.) again uses horizontal lines as aids in multiplication and division of fractions; thus the division of $\frac{3}{4}$ by $\frac{2}{3}$ is written $\frac{3}{4} \div \frac{2}{3}$.

Cajori (p. 260.) refers to the use of $\times$ in division of fractions by Steinmetz (No. 86, f. 68, r.) in his 1568 edition. In 1575 Steinmetz uses an additional $\times$ when he divides $\frac{3}{4}$ by $\frac{2}{3}$ in this manner: $\frac{3}{4} \times \frac{3}{2} \times \frac{9}{8}$. The scheme used by Manelli (No. 59, p. 33.) is different, for he writes the divisor first and then multiplies along the heavier bar of the capital $\times$ first; thus, in dividing $9\frac{1}{2}$ by $6\frac{2}{3}$ he writes:

$$\begin{array}{r} \frac{20}{3} \times \frac{19}{2} \\ 40 \overline{) 57} \quad 17 \\ \underline{40} \quad 17 \end{array}$$

It is of interest to note the mnemonic signs for operations with fractions used by Graffenried (No. 42, p. 116.), in which $\times$ denotes a scheme for addition and subtraction of fractions, $=$ for multiplication, and $\div$ for division. This same scheme was used by Apian in 1527 (See P. Treutlein, "Das Rechnen im 16. Jahrhundert", Abhandlungen zur Geschichte der Mathematik, v. 1, p. 80.).

12. Casting out the 9's. In many early arithmetics various settings of lines afforded a convenient grouping of the four results of an operation. The characteristic $\times$ and $\times$ were again used by Clavius in casting out 9's, although Cajori (p. 262.) simply refers to his use of $\times$ for this purpose. Rudolff (No. 81, f. 12, r.) uses a small $\times$ in this manner: $8\frac{4}{5}$. In Regius (No. 76, f. 56.) we find a small $\times$, but Micyllus (No. 60, p. 9.) uses the usual large $\times$. Suevus (No. 88, p. 52.) checks his multiplication only by

casting out 9's and uses $\times$ for this purpose. We have referred to Savonne's (No. 82, p. 52.) use of the Greek and Latin cross in casting out 9's in his 1571 and 1619 editions, respectively. It is strange, then, that he uses the usual $\times$ in addition and subtraction, but no guiding lines of any kind for multiplication and division of fractions.

13. Multiplication of integers. In multiplication 'per crocetta', the product of two numbers is found mentally with the aid of guiding lines. Bassi (No. 5, p. 26.) uses this setting:

$$\begin{array}{r} 4 \quad \times \quad 15 \\ 1 \quad \times \quad 1 \\ \hline 3 \quad \times \quad 6 \\ \hline 1620 \end{array}$$

Figatelli (1678) and Manelli (No. 59, p. 14.) use a similar procedure.

14. To mark the place for 'thousands.' The use of $\times$ marked on the line on which a dot signified '1000' in computations upon lines is found in early arithmetics, as in those of Feme (No. 31.), Rudolff (No. 81, f. 30, v.), Regius, (No. 76, ff. 96, r., 61, v.) and Sesen (No. 84, p. 5.). Reisch (No. 77, f. 94.) also puts a cross at the seventh or million-line, when the latter occurs. (See F. P. Barnard, The Casting-Counter and the Counting Board, Oxford, 1916, p. 271.).

15. In place of multiplication table above 5 x 5. Cajori (p. 264.) mentions the use of $\times$ in the so-called complementary multiplication. He refers to Regius (1543), fol. 56, whereas it should be fol. 61. The $\times$ is also used by Regius in a further complementary method of finding the product of two integers, say 8 and 7, thus:

$$\begin{array}{r} 8 \times 2 \\ 7 \times 0 \\ \hline 14 \\ 56 \end{array}$$

In this process he multiplies the complement of the larger number by the smaller number, and subtracts this result from the product of the smaller number and ten. Blondel (No. 9, p. 84.) illustrates the product of the same two numbers in the usual manner:

$$\begin{array}{r} 8 \times 2 \\ 7 \times 3 \\ \hline 56 \end{array}$$

Micyllus (No. 60, p. 17.) also gives an illustration of the product of numbers above 5, but Steinmetz (No. 86, f. 28, r.) and Monier (No. 62, p. 55.) both present the same problem in which one of the numbers is less than five, thus offering an added difficulty; for instance:

$$\begin{array}{r} 8 \times 2 \\ 1 \times 1 \\ 4 \times 6 \\ \hline 12 \\ 2 \\ \hline 32 \end{array}$$

16. The St. Andrew's cross used as a symbol of multiplication. The St. Andrew's cross $\times$ used for this purpose assumed various forms in its early history, after its first use by Oughtred (1631). We find large and small-sized symbols, $\times$, $\times$, $\times$, $\times$, a small elevated $\times$ as in Moore (1660). In solving the proportion $HM:IK::BE:ED$, Ceva (No. 20, p. 12.) uses $\times$ when he writes $HM \times ED = IK \times BE$. However, we do not find $\times$ commonly used as a sign of multiplication in the texts of the seventeenth century.

4. SIGNS FOR DIVISION

17. Early symbols. In the texts of the sixteenth and seventeenth centuries, division is most commonly expressed by separating the dividend from the divisor by such words as per, por, a, by, or durch. In illustrations, however, one or two curved bars, vertical and horizontal bars are used as symbolisms of division. The two signs of division, $\div$ and $:$, accredited to Rahn (1659) and Leibniz (1684) respectively, did not become common until the beginning of the eighteenth century.

18. Relative position of divisor and dividend. In these early books the divisor and quotient have been assigned various positions relative to the dividend. Thus below we cite eight authors (seven not in Cajori) whose works present eight different positions of divisor and quotient relative to the dividend. These variations are not noted by Cajori.

We begin with the arrangement most common for the 'scratch method' of division. In this case the divisor is placed below the dividend, and the quotient is usually placed to the right of the dividend. This, as noted above, was the case with Husswirt (1501), Savonne (1571, 1619), and Kamus (1599). Waser (No. 99, p. 26.), Bassi (No. 5, p. 58.), Puig (No. 72, p. 75.), Prestet (No. 70, p. 30.), and Figatelli (No. 32, p. 30.) (who are somewhat representative of writers in Latin, Italian, Spanish, French, and Italian respectively), follow a similar practice, placing a vertical bar, or curved bar, or some variation of it, to the right of the dividend. Griminelli (No. 43, p. 67.) places the quotient to the left of the dividend in this method.

In the 'a Danda' method of division, as our method was called, we find various arrangements. The most common can

be represented best as $a|b(c$, the mode followed by Bassi (No. 5, p. 55.), Bonvicino (No. 12, pp. 11-14), and Figgatelli (No. 32, p. 30.), who use vertical bars as symbolisms for division instead of curved bars. Bonvicino (No. 12, p. 12.) presents an arrangement in short division in which the quotient is above the dividend, something like the following:

2328

lowing: $3|6984$. Venturoli (No. 94, p. 7.) differs by having the quotient below the dividend as in $743|978531$

1317.

Still another variation is found in Griminelli (No. 43, p. 63), who places the quotient below the divisor in this manner:

647 546535.
844

A more uncommon practice is also given by Bassi (No. 5, p. 61.), who places the divisor above the dividend. This same method has been noted in Cataneo (1546), whereas Sfortunati (1561) placed the divisor either above or below the dividend. The very peculiar arrangement given by Finé of placing the quotient between the dividend and divisor, as given p. 15, is noteworthy. Another instance of the so-called French practice of writing the divisor on the right is found in the Spanish Puig (No. 72, p. 82.), who writes:

$$\begin{array}{r} 1706549 \overline{)458} \\ 3726 \end{array}$$

5. SIGNS OF PROPORTION

19. Arithmetical and geometrical progression. Oughtred's notation for continued geometrical proportion was not commonly used until the latter part of the seventeenth century. We find only a few instances of its application in the texts under consideration. Cajori (p. 279.) has referred to the use of $\div$ by Sir Jonas Moore in his 1688 Arithmetick; Moore's use of this symbol as early as 1660 has been cited previously. Isaac Barrow, according to Cajori, adopted Oughtred's symbol in his Latin (1655) and English (1660) editions of Euclid. We find this same symbol in Barrow's later work of 1675 (Archimedis Opera Conicorum, Libri III, London, p. 31.). In the same year, the French Prestet (No. 70, pp. 188, 170.) uses $\div$ for geometrical progression, writing $\div 2.4.8.16.32.64.128.$, etc. (This symbol is still sometimes seen in French textbooks. See D. E. Smith, History of Mathematics, v. 2, p. 413.) For arithmetical progression, however, he

uses no symbol preceding the numbers, merely dots between them, as 1.3.5.7.9.11., etc.

Symbols for arithmetical progression are not in common use in the early books. The sign $\div$ for arithmetical progression was not extensively applied until the eighteenth century. Ordinarily, for both kinds of progressions, we find a variety of symbols separating the numbers, as dots, commas, colons, or vertical bars, but no special symbol distinguishes arithmetical and geometrical progressions.

The use of dots simply as signs of separation has been mentioned in connection with Husswirt (1501). Followers of this practice are many: Noviomagus (No. 65, f. 43, v.), Vuelpius (No. 98, f. 38, v.), Micyllus (No. 60, p. 45.), Steinmetz (No. 86, f. 45.), Waser (No. 99, pp. 43, 45.), Clavius (No. 25, pp. 228, 243.), Alsted (No. 2, pp. 53, 62.), Bassi (No. 6, pp. 504, 505.), Puig (No. 72, pp. 267, 268.), Monier (No. 63, pp. 431, 436.), and Lucas (No. 58. pp. 180, 185.). Manelli (No. 59, p. 16.) writes 1.e3.e5.e7.e9, etc. and 1.e2.e4.e8.el6., etc. Commas are used by Gemma (1549) and Peverone (No. 68, p. 20.). Blondel (No. 9, pp. 225, 229, 230.) uses commas or colons, and Unicornio (No. 93, p. 68.), vertical bars.

20. Arithmetical proportion is not so commonly mentioned in the early texts, and a special notation for it is still less common. Just as in progressions, dots are used merely between the terms of the proportion. Alsted (No. 2, p. 53.) illustrates a 'disjunct arithmetical proportion' by 10.8.4.2. In Prestet (No. 70, p. 169.) there is a theorem on arithmetical proportion and the problem of finding the fourth term of the arithmetical proportion 15.10.18 as 13.

21. Geometrical proportion. The 'Rule of Three' or of 'Proportion', as it is commonly called, was very popular in the early arithmetics, and we find various methods of presenting the numbers involved in a geometrical proportion. Short words were often used to write the proportion. Noviomagus (No. 83, p. 136.) writes: 10 efficiunt 7. quot 17? 11.9. In a similar way, Scheubel (No. 83, p. 136.), in the

10

application of the rule of three, uses the form:

Res aureis		Res
21	18	quanti 29,

and Clavius (No. 25, p. 118.) writes:

scudi	Libre	Scudi	Libre.
4	12.	20?	fanno 60

Sometimes horizontal lines are used to separate the three terms in the proportion, as by Alsted (No. 2, p. 55.), Lambeck (No. 55, p. 45.), Bassi (No. 6, p. 78.), Venturoli (No. 94, p. 81.), Behme (No. 7, p. 52.), and Blondel (No. 9, p. 177.). Rudolff (No. 81, f. 39, r.) separates them by dots and writes 3. 2. 12. for 3:2::12:x. Blondel also uses the colon in proportions as well as in progressions, thus: 2:4:5:10, but Vuelpius (No. 98, f. 43, v.) simply separates the terms by spacing. As late as 1697, Lucas (No. 58, p. 84.) as *Finé* in the 1587 edition, designates the terms of a proportion by A, B, C, and D, respectively, thus

A.2	B.6.	C.8.
		B.6.
		2 48.E
		flor. 24.D

In addition to the proportion notations given by Cajori, there is the use of the inclined bar by *Finé* in 1555, as well as Figatelli's clumsy notation for proportion, consisting of braces and words (illustrated above). Cajori refers (p. 289.) to a notation by Herigone (No. 49.) in which the letter π signifies ratio, but he does not mention that, in the explanation of the notes, Herigone writes the proportion $a:b::c:d$ in these two ways: $a \pi b \ 2|2 \ c \pi d$ or $a \pi |b$
 $c \pi |d$.

(Cajori incorrectly says on page 289 that the first edition appeared in 1642, and on page 277 he refers to an 1834 edition, meaning 1634.)

22. Oughtred's and Wing's notations. The dot for geometrical ratio, and the four dots of proportion, $::::$, introduced by Oughtred (*Clavis Mathematicae*, London, 1631.), were used by Sir Jonas Moore (1660). This symbol, and a modification of it in the form $a :: b:b \div X$, were employed by Alexander (1693), noted by Cajori and above (p. 24.) Prestet (No. 70, p. 187.) is again a follower of Oughtred in the use of the notation $::::$ for geometrical proportion. It is of interest to add that, in using this notation, Prestet says, after writing $\frac{6}{3} = \frac{8}{4}$, that he is 'persuaded that this sort of expression, 6.3::8.4, has been chosen to avoid that of small characters which mark fractions, and in order not to use the sign of equality.'

The colon as the symbol for ratio, introduced by Vincent Wing (1651), was not generally adopted before 1700. Ceva (No. 20, p. 12.) uses it, but instances of the use of either Oughtred's or Wing's notations in the late seventeenth-century arithmetics are very rare.

6. SIGNS OF EQUALITY

23. Equality expressed by words and abbreviations. For more than a century after the introduction of the symbol $=$ by Recorde in 1557, equality was still designated by words and abbreviations. In the German and Latin works of this early period, we find such expressions as facit, faciunt, aequales, and aequantur, among the authors Rudolff (1532), Cardan (1537), Regius (1543), Noviomagus (1544), Micyllus (1555), Scheubel (1560), Suevus (1593), Waser (1603), Vieta (1646), Behme (1695), and Lucas (1697). The Italian eguale à is found in Bombelli (1579) è in Griminelli (1670), and fanno or fà in Unicornio (1598), Zuchetta (1600), Manelli (1659), and in the Italian edition of Clavius (1618). Aurel (1552) and Puig (1672) give the Spanish ygal a, igual à or ig. à respectively. In the French works we find the words égal and font as in Vieta (1630) and Blondel (1699).

24. Recorde's sign of equality. Cajori (p. 303.) cites the Swiss Johann Alexander as employing Descartes' sign of equality ∞ ; in reality, we find that although Alexander gives both $=$ and ∞ as equality symbols on page two of his 1693 edition, he uses Recorde's symbol $=$ throughout his work in problems. Harriot (1631) and Prestet (1675), by adopting Recorde's symbol, did much to give it a strong foothold in England and France.

In Italy, Recorde's symbol is used by Renaldini (No. 78, pp. 85, 86.) in 1665. Cajori does not include any mention of the early use of this symbol in Italy, although Renaldini is later listed by Cajori among the representative Italian writers. In Renaldini's work, the type used in printing the equality symbol is the letter l, placed horizontally, thus = ; the l's were sometimes shorter and closer together as = . This form of the symbol was very convenient, for one of the l's generally served as his minus sign.

Our modern form of Recorde's sign of equality $=$ is used also by the Spanish Ventura de Avila (No. 4, p. 2.), whose work is in dialog form much like Recorde's Ground of Artes.

Ceva's symbol (No. 20, p. 9.) consists of very long lines close together.

25. Symbols for equality before 1700. The majority of writers of the sixteenth century expressed equality by words, as noted above. Even in the seventeenth century, we find noted mathematicians using no regular symbol for equality. Recorde's sign, if any, was used by writers in the second half of the seventeenth century, but this sign was not practically established until the beginning of the eighteenth century.

7. SIGNS OF COMMON FRACTIONS

26. The fractional line was generally used by sixteenth and seventeenth century authors although there are several instances of its omission in the case of 'fractions of fractions.' Cajori's reference (pp. 152, 153.) to the omission of the fractional line after the first fraction by Clavius in his *Epitome arithmeticae* was to the 1583 edition. We also note this same feature in his 1585 and 1586 editions, usually in the case of the more common fractions. No such omission is made, however, in his later works of 1618 and 1686. Tunstall (1538), Finé (1555) and Ramus (1599), as noted above, also have this distinct notation for 'fractions of fractions', and Savonne (1619) frequently omits the fractional lines altogether in the fundamental operations. We find that Ghaligai (No. 38, pp. 15, 17.) sometimes writes his fractions without a separating line, but since this is not usual, it may possibly be the result of faintness in type.

Broken fractional lines, so very common in these early centuries, are an indication of the difficulty experienced in printing fractions with large numerators and denominators. We have remarked this concerning Husswirt (1501) and Peurbach (1539), and we may add Clavius in all his editions, Regius (1543), Noviomagus (1544), Tunstall (1544), Feliciano (1550, 1692), Gemma (1550), Micyllus (1555), Tartaglia (1556), Sfortunati (1561), Steinmetz (1575), Jacob (1612), Ramus (1613), Napier (1623), Chauvet (1627), Girard (1629), Alsted (1641), Moore (1660), Hemeling (1667), and Rahn (1668).

The length of the fractional line was not standardized, as may be seen in the case of Renaldini (No. 78, pp. 447, 296.), who generally uses a very long line, but sometimes

we find various lengths of lines on the same page. Renaldini leaves considerable space between the fractional line and the terms of a fraction. This feature is also noted in the early and rare German book of Felner (No. 30, f. 26.).

An extraordinary fractional notation is brought out as late as 1675 in the work of Ciacchi (No. 22, pp. 48-50.). In his arithmetic he generally employs fractions whose terms consist of only one digit, and these numbers are placed sidewise in the running text, thus: $\overline{3}$ or $\overline{4}$ and $\overline{5}$ or $\overline{6}$. Somehow he prefers $\overline{3}$ and $\overline{5}$ to the alternative forms, probably because these can be written and read with greater facility and naturalness. In illustrations, however, he sometimes uses a heavy horizontal line, and writes the numerator and denominator very far apart (*Ibid.*, p. 58.). In case of fractional terms of more than one digit he uses the ordinary form, as in $\frac{232}{299}$ (*Ibid.*, p. 182.).

8. SIGNS OF DECIMAL FRACTIONS

27. Early notations. Many of the early writers, in the division of numbers by 10, used a vertical bar to separate the quotient from the remainder. We might interpret this symbol as a decimal separatrix, but these writers themselves did not have the full concept of decimal fractions. (Rudolff's Exempelbüchlin, Augsburg, 1530, is the only known case of the use of the decimal bar at that date. See D. E. Smith's article in Teachers College Bulletin, s. 1, v. 5, 1910-11, p. 21, in which Rudolff's discovery was first announced. Eneström, however, thinks that Rudolff did not realize the full significance of his work. See Scripta Mathematica, Dec., 1932, pp. 168, 169). In addition to the writers mentioned by Cajori we have noted above the use of the vertical stroke by Tagliente (1520), Wolphius (1534), Cataneo (1546), and Sfortunati (1561), all of which appeared before the first printed explanation of decimal fractions by Stevin in 1585 (See D. E. Smith's Source Book in Mathematics, New York, 1929, for a selection from "Stevin on Decimal Fractions", pp. 20-34). Ramus (1599) also made this close approach to decimal fractions.

We quote from Rudolff's Künstliche rechnung (No. 81, f. 10, v.):

Zu mercken das 1 weder multiplicirt noch dividirt.
Demnach wañ ein zal geteilt sol werden in 10/ schneid
ir ab mit einer virgel die erst ziffer/ die figuren
gegen der lincken hand sein der quocient / uñ die
erst abgeschnitten figuren das rest.....

Geteilt in 10	Geteilt in 100	Geteilt in 1000
678 5	67 85	6 785

Similarly, Felner (No. 30, f. 15, v.) divides 67845 by 100 thus:

$$\begin{array}{r|l} 678 & 45 \text{ facit } 678 \\ 1 & 00 \end{array}$$

das Rest 45. macht
am kleinsten $\frac{9}{20}$

The figure 45 looks like $\frac{45}{100}$, but Felner regards it simply as a remainder. Tartaglia (No. 90, f. 30, v.), Scheubel (No. 83, p. 52.), Steinmetz (No. 86, f. 43, v.), and Unicornio (No. 93, p. 38.) also use the vertical bar in illustrating division by 10 and 100. Peverone's (No. 68, p. 18.) separatrix takes the form of an inclined bar but serves the same purpose. Even as late as 1659 we find Manelli (No. 59, p. 17.) employing the notation $87\overline{10}$ in the division of 870 by 10, a close approach to the rectangular line symbol $\perp$ used by Oughtred (1631) in the writing of decimals.

28. Seventeenth-century notations. An illustration of use of the decimal comma and point as separators of units and tenths by Napier in his Raddologia of 1617 is given by Cajori (p. 323.). In it the quotient of 861094 and 432 is written as 1993, 273. In the 1623 edition of the Raddologia, which is an Italian translation of the 1617 edition, we find this same quotient written as 1993. $\overset{2}{\underset{7}{\underset{3}{\text{'''}}}}$ (No. 64, p. 56.). Again, Cajori (p. 324.) states that Napier writes 1994,9160 for 1994.9160, but in the 1623 edition he merely uses the comma, as 1994,9160 (Ibid., p. 94.). Another use of the comma for the same purpose occurs when Napier writes 821,25 for 821.4 (Ibid., p. 88.). These three examples indicate his vacillation in the use of period, comma, and sexagesimal exponents as decimal symbols.

Reference has been made to Sir Jonas Moore's five different decimal notations (see p. 22.), whereas Cajori notes for this author only one symbol, the inverted comma. Voigtel (No. 97, p. 2.) devotes the first chapter of his *Marckscheide-Kunst* to 'Decimal-Rechnung', and in it he has a preference for the sexagesimal exponents for decimals; he usually writes $365^{\circ}735''$, occasionally $5.8.9.2$. In Lucas (No. 58, p. 225.) there is a chapter of eight pages on 'Arithmetica geometrica decimalis'; in it he uses the tau-tological notation $11.4.5.6$. Much earlier, Graffenried (No. 42, p. 648.) discusses decimals on seven pages and writes them in the form $3|587$.

An entire text devoted to the doctrine of decimals, including tables of interest and rebate, is that of Hodder (No. 50, pp. 12, 22.). The comma is given as a symbol once, but the period is otherwise used throughout. The decimal 0.35 is read as 3 Primes and 5 Seconds according to custom. This text is of particular interest as Hodder's was the first separate treatise on arithmetic published in English in America (1719).

9. SIGNS OF POWERS

29. Early symbolisms. In the early Latin books, in connection with the extraction of roots, one often finds the expressions numerus quadratus and numerus cubicus to represent the square or cube of a given number, as in Vuelpius (No. 98, f. 57, r.). Micyllus (No. 60, p. 194.) labels the 4 and 8 of the geometrical progression 1, 2, 4, 8, .. as quad., Cub. respectively. Very frequently a table of the squares and cubes of numbers is given with the column headings R., Q., C.; we find this in Bonvicino (No. 12, p. 33.) as late as 1665. The powers of known numbers did not need any symbolism until the latter came to be represented by letters; there was need, however, of a symbolism for powers of the unknown.

30. Additive principle. The additive principle of the Greeks in combining powers was not so popular as the multiplicative principle of the Hindus. Cajori (p. 342.) has referred to the use of the additive principle by Vieta in several of his publications, but not to his important French work, L'Algebre, of 1630. In the introduction Vieta (No. 95, pp. 7, 8.) lists the names of the powers according to the

additive principle as follows: Le costé ou la racine, le quarré, le cube, le quarré quarré, le quarré cube, le cube cube, etc. In the second book of 'Zeteticques' (Ibid., pp. 80, 81.) he uses the symbols N, Q, and C for the unknown number and its powers; in the third book, the abbreviations are Aq., Ac.

Another French writer, Blondel (No. 9, p. 18.), uses the additive principle, but he has a preference for capital letters thus: R, Q, C, QQ, CC, etc.

31. Multiplicative principle. Pacioli has many followers in the use of his power symbols, co., ce., cu., ce.ce., p.r., ce.cu., etc. Feliciano's use of these Italian symbols in the 1550 edition has been noted above, but in 1669 the editor of his work substituted A, AA, etc., for them. Unicornio (No. 93, p. 136.) uses cosa, censo, cubo, ce.ce., l.rel., ce.cu., etc., merely a slight modification of Pacioli. In Bassi (No. 6, p. 545.) we find numero, cos., quad., cub., quad.qu., prim.rel., quad.cub., the only variation of note being qu. for ce. These same forms with more intense abbreviations were noted in Figatelli (1678). The Spanish Puig (No. 72, p. 369.) capitalizes the symbols of Pacioli, thus: N., Co., Ce., Cu., Cce., lR., CeCu.

The German symbols, ϕ (dragma or numerus), χ (radix), ζ (zensus), κ (cubus), $\zeta\zeta$ (zensdezens), β (sursolidum), $\zeta\kappa$ (zensicubus), as noted above, may be found in Jacob (1612) when he writes rad., zens., cub., zensdezens, but uses no further abbreviations. The same symbols are used by the Dutch writers, Sesen (No. 84, p. 654.), Coutereels (No. 26, p. 334.) and Hansz (No. 44.), and by many others. Lambeck (No. 55. p. 299.) uses in the Appendix these symbols of Rudolff with slight differences for dragma, radix, and cubus, thus: Dr., rr., c ζ ., cs., $\zeta\zeta.$, $\beta.$, ζ cs. We find these symbols in Alsted (No. 2, p. 69.) with some alternate terms: cubus, biquadratus and zensizensus, solidus, quadraticubus, zensicubus. In a chapter on 'Algebra vulgaris', Lucas (No. 58. p. 285.) uses this curious combination of symbols, N, R, Q, C, bq, Ss, cq.

Thus we see that even a half century after Descartes published his Geometrie (1637), authors were still adhering to the old notations of the 'Abbreviate Plan', using either the additive principle, or the multiplicative principle with its German or Italian symbolism. The so-called 'Index Plan' was a step towards our modern notation, and we find various authors developing and perfecting this plan. Cajori (p.243.)

gives attention to a new symbolism of crossed out numerals as exponents used by Pietro Cataldi in an algebra of 1610. The present writer also finds limited examples of the 'crossed out' numeral exponents, as $1.\cancel{x}$ and $1.\cancel{\beta}$ for X and X^3 respectively, in the 1619 work (No. 17, p. 2.), for Cataldi reverts to the Italian symbols after page 7.

32. New exponential notations. The Cartesian notation became generally adopted in England and in Continental Europe during the second half of the seventeenth century. We are able to give additional references for this notation to the English and French works given by Cajori, and also to German, Swiss, and Spanish works.

The exponential symbolisms employed by Sir Jonas Moore as early as 1660 are passed over in silence by Cajori. Previous reference has been made to his three different notations, $\underline{Ac}$, $\underline{aaa}$, and $\underline{a^3}$, according to Oughtred, Harriot, and Descartes, respectively. In the last few pages of Lambeck's Rechenbuch (No. 55, p. 308.), in which Johann Wohlgemuth adds solutions of six algebraic problems, our modern exponents occur, except that he has no preference for $\underline{xx}$ or $\underline{x^2}$.

An early trace of the Cartesian notation in France is found in Jean Prestet (No. 70, pp. 65, 66.), who follows Descartes also in the writing of $\underline{aa}$ for $\underline{a^2}$. The algebra of the Swiss John Alexander in 1693, as noted above, designates involution by means of exponents rather than with the Archimedean spiral used by Rahn in 1659.

At Barcelona, the Spanish Ventura de Avila (No. 4, pp. 69, 95, 103.) brought out an algebra in which the equations contain the unknowns $\underline{m}$, $\underline{n}$, and $\underline{t}$ with their integral exponents in the modern form, with only an occasional $\underline{nn}$ for $\underline{n^2}$. The Italians, outside of Renaldini (1665), do not seem to adopt the new exponential notation so early, probably due mostly to the very great influence of Pacioli and his followers, and to some extent, to the novel notations of Ghaligai, Bombelli, and Cataldi.

10. SIGNS FOR ROOTS

33. Early expressions for roots. In the extraction of square and cube roots, common in the early arithmetics, such expressions as radix quadrata, radix cubica, and their equivalents, were sufficient and needed no symbolism. These

terms may be found in the works of Vuelpius (1544), Micylus (1555), Steinmetz (1575), Waser (1603), and Lambeck (c. 1661). Late into the seventeenth century, Griminelli (1670), Ciacchi (1674), and Monier (1693) were satisfied with these expressions. Even such prominent writers as Scheubel (1560), Clavius (1586), and Napier (1603) used no further notation for radical expressions.

34. The basic symbol R. As noted above, Jacob (1612) simply expressed the second, third, and fourth roots by radix quadrata, radix cubica, and radix zensdezens, respectively. Unicornio (No. 93, ff. 126 v. 127.) again follows Pacioli in his manner of designating the order of the roots, but uses the small letter r for radix, thus: r.7, r.cu.7, r.r.5, r.rel 5, r.cu.q.9. A variety of notations is contained in Bassi (No. 6, pp. 530, 541, 542.). He uses rad. quad., rad. cub., rad. qu. qu., rad. rel., rad.qu.cub., later R, R, or Rad. In Puig (No. 72, p. 484.) we find rq., rc., and rrq. As late as 1697, Lucas (No. 58, p. 319.) gives this unusual notation: radix Q, radix C., radix bq., radix ss., radix CQ., radix ssb., radix triq., and radix CC.

The R, used by Pacioli as a fundamental symbol for indicating roots, reached its zenith in Italy during the sixteenth century. It was used by Feliciano in 1550, and still survived in Figatelli in 1678. In 1619, Cataldi (No. 17, p. 5.) follows the general symbolism R, Rc., Bq.c., for square, cube, and sixth roots respectively. We have found an instance of the use of the crossed capital R in Vieta (No. 95, pp. 183, 184.), who used l (latus), or the word radix, for indicating roots. We must mention Renaldini (No. 78, p. 21.), who represented square root by RQ or simply R, cube root by RC.

35. The basic symbol √. Sir Jonas Moore (1660), as noted above, marked the order of the radical as did Oughtred, √, √c, √qq. Just as many authors used the sign □ as an operator to indicate squaring, Lambeck (No. 55, App. pp. 308, 309.) used it in the extraction of square root; he writes √□, and √cub.

In France, Prestet and Blondel used √ extensively, differing only in the mode of designating the order of the root. Prestet (No. 70, p. 108.) expresses square roots simply by √, but he has two forms for cube root, √C. and √3^e; however, he gives preference to the former. Likewise fifth and

seventh roots are $\sqrt[5]{5^e}$, and $\sqrt[7]{7^e}$, and in sixth and tenth roots he shows the commutative order of the radicals when he writes $\sqrt{C}.\sqrt{}$ or $\sqrt{}C.$, and $\sqrt[5]{5^e}\sqrt{}$ or $\sqrt{}\sqrt[5]{5^e}$, respectively. Fourth, eighth, and ninth roots similarly follow the multiplicative principle.

Blondel (No. 9, pp. 18-20.), however, is consistent when he uses the additive principle in his notation for roots as well as for powers, for he says in French that 'the roots derive their name from the relation to the power to which each of them is compared.' For expressing roots, he uses his notation for corresponding powers preceded by the radical sign, thus: $\sqrt{}, \sqrt{C}, \sqrt{QQ}, \sqrt{QC}, \sqrt{CC}, \sqrt{QQC}$. It is noteworthy that he gives as alternate forms the symbolism used by John Wallis in 1655, in which the root indices in numerals are elevated and placed after the sign $\sqrt{}$ in this manner, $\sqrt^3{}, \sqrt^4{}, \sqrt^5{}$, etc. The Dutch writers, Sesen (No. 84, p. 667.), Coutereels (No. 26, p. 429.), and Hansz (No. 44.), use instead the German symbols to mark the order of the root.

The Swiss Alexander (1693), as noted before, represents roots as did Moore, but he follows the letters by dots. In Spain, Ventura de Avila (No. 4, pp. 99, 102, 103.) uses the modern sign for square root on a few pages. It is interesting to note in passing that in solving the problem $nn + 13 = 6n$, he gets the roots $n = 3 + \sqrt{-4}$ and $n = 3 - \sqrt{-4}$. He proceeds to say that since these roots are impossible, the problem is impossible.

11. SIGNS FOR UNKNOWN NUMBERS

36. Early forms. In our survey of powers of the unknown we met with the term cosa as the Italian word for the first power of the unknown x . Cosa, or some abbreviated form of it, is used by Feliciano (1550), Unicornio (1598), Bassi (1666), Puig (1672), and Figatelli (1678).

Radix, or some equivalent symbol, is found in Jacob (1612), Graffenried (1618), Bonvicino (1665), Lambeck (c. 1661), Hemeling (1667), Lucas (1697), and Blondel (1699).

Cajori (p. 380.) has referred to the use of N (numerus) for x by Vieta in 1615; Vieta further uses also I N in the 1630 edition (French). As noted above, Feliciano changed from the Italian symbolism to that of Vieta (1591) when, in 1669, he wrote A for the unknown. Likewise Moore (1660) uses the large or small characters, A or a, for x . Cataldi, as noted, exhibits the notation of the crossed 'one' for x ,

but Cajori (p. 344.) says he made only very limited use of this notation. It has been pointed out previously that in 1619, after only a few pages of the use of these 'crossed out' numeral exponents, Cataldi resumes the Italian symbol co. for the unknown.

A still different notation appears in the algebra of the Spanish Ventura de Avila (No. 4, p. 64.). He speaks there of the 'tres incognitas, n, m, t,' and uses them throughout his work. He makes no explanation for his use of m and t as the second and third unknowns.

37. Descartes' z, y, x. A prominent early follower of Descartes' way of signifying known quantities by the first letters of the alphabet, and unknowns by the last, z, y, x, is the French writer Prestet (No. 70, p. 150.). He says:

pour les autres grandeurs que l'on ne connoît pas
encore, elles seront exprimées par les dernieres let-
res z, y, x, v, &c.

In Brancker's translation of Rahn's Algebra (No. 73.), the x, y, and z are also used freely. Thus the Cartesian notation for the unknown was being adopted in France in the late seventeenth century in purely algebraic works (also in Ozanam's Nouveaux Elemens d'Algebre, Amsterdam, 1702; see Cajori, Notations, p. 370.) as well as in editions of Descartes' geometry.

12. SIGNS OF AGGREGATION

38. Aggregation expressed by letters. Although Cajori makes the statement (p. 385.) that the use of letters for aggregation practically disappeared in the seventeenth century, we have on hand several noteworthy instances of their use in Italian and Spanish arithmetics. Unicornio (No. 93, f. 142, v.) uses the abbreviations rad., v. and rad. legata in marking the radix universale (aggregate) and radix legata (single term), for he writes rad. v. 5, p. rad. v. 3, p. rad. legata 49, p. rad. 16, rad. 4 for our

$\sqrt{5} + \sqrt{3} + \sqrt{49} + \sqrt{16} + \sqrt{4}$. Following Bombelli, the inverted capital letter L was employed by Cataldi (No. 17, p. 13.) to express aggregation. Thus he writes: Rc. $\left[7\frac{1}{2} \text{ } \overline{\text{L}} \text{ } 52 \text{ } \overline{\text{L}} \right]$
for $\sqrt[3]{7\frac{1}{2} - \sqrt{52} \frac{1}{12}}$. Cajori (p. 135.), however, mentions

Cotaldi's use of the dot to express aggregation.

The Spanish Puig (No. 72, p. 484.) explains the symbols rv., rcv., and rrv. as indicating 'raiz quadrada universal, raiz cubica universal,' and 'raiz quadrada de raiz quadrada universal.' It is important to note Figatelli's use of RV to signify universal root, the root of a polynomial, which he curiously follows by the left part of a parenthesis, and thus: RV (12. più R 5 for $\sqrt{12 + \sqrt{5}}$ (No. 32, p. 195.).

39. Aggregation expressed by the vinculum or bar above. As does Cajori, we also find the vinculum adopted by J. Prestet (No. 70, pp. 108, 136, 121.), but it is important to add that in connection with the vinculum and radical sign he sometimes uses a dot in cases like the following: $\sqrt{\cdot \sqrt{5} + \sqrt{3}}$ and $\sqrt{\cdot \sqrt{1\frac{1}{4}} + \sqrt{\frac{1}{2}}}$ for $\sqrt{\sqrt{5} + \sqrt{3}}$ and $\sqrt{\sqrt{1\frac{1}{4}} + \sqrt{\frac{1}{2}}}$ respectively. However, he does not frequently have occasion to use this combination of symbols, but he is very liberal in using the vinculum to indicate aggregation in the treatment of radicals as well as in its general use in such expressions as $\overline{a+b} \sqrt{a+b}$.

Mention has been made of the use of the horizontal bar in connection with the radical sign by John Alexander (1693). Ventura de Avila (No. 4, pp. 60, 106.) is sparing in his general use of the vinculum; occasionally such expressions

as $3 \times \overline{8m - 6} = m$ and $\frac{7v + 60}{3} \times \frac{v - 3}{3}$ occur.

40. Aggregation expressed by parentheses. Renaldini (No. 78, 79.) is listed by Cajori (p. 392.) among the Italian writers using parentheses. In the Ars analytica mathematicum (No. 78, pp. 21, 317.), to which all of Cajori's references to Renaldini are made, this statement is given:

Signum autem hoc R (), significat radicem universalem, sive ligatam; itaq; BQ () denotabit radicem quadratam universalem,...

However, in his Artis analyticae mathematicum (No. 79, p. 96.) is the explanation:

Quod si in animo, sit extrahere radicem ex binomio, ut ex $a^2 \uparrow b^2$,..., qui sequitur in modum R ($a^2 \uparrow b^2$), vel loco parentesis adhiberi etiam solet lineola desuper ut R $a^2 \uparrow b^2$.

Thus in the second volume Renaldini employs parentheses and vinculum interchangeably. Occasionally also, in the first part, brackets are employed as aggregation symbols, sometimes brackets and parentheses on the same page. These same symbols are also used to enclose parenthetical statements.

It is worthy of note that the second volume of Renaldini is an additional source for the use of the usual star to mark the absence of terms, as in this expression:

$$a\bar{3} \uparrow ba^2* - \bar{z}^3.$$

III

COMPLETION OF CERTAIN TOPICS IN TROPFKE

A. NUMBERS IN GENERAL

1. Numeral Words

In numeration the word for one thousand is commonly mille (Latin), mil (French), mille (Italian), tausend (German), and thousand (English) in the texts of the various countries during the sixteenth and seventeenth centuries. Numbers higher than one thousand are named largely from their formation by the multiplicative principle; thus ten thousand may be found expressed as decem millia, and one hundred thousand as centena millia.

In some early Latin books 10^6 (million) is occasionally called decies centena millia, as by Regius (No. 76, f. 51, v.) and Steinmetz (No. 86, f. 12, v.); but a form corresponding to mille millia is used by Noviomagus (No. 65, f. 23, r.) and Vuelpius (No. 98, f. 8, r.) and millena millia by Waser (No. 99, f. 10, v.) and Alsted (No. 2, p. 25.). Tausend mal tausend was general in Germany until the word million appeared in the arithmetic of Christoff Rudolff (1526). (See Tropfke, v. 3, p. 7.). We also find tausend mal tausend oder million mentioned in Rudolff's 1532 arithmetic (No. 81, f. 3, r.), but in reading of a large number he uses tausendmal tausend. About the same time the rare arithmetic of Felner (No. 30, f. 4, r.) gives only tausent mal tausent. Suevus (1593), in devoting nineteen pages to numeration, with his customary prolixity, does not once use the word million, nor does Behme as late as 1695. (According to L. L. Jackson, Sixteenth Century Arithmetic, New York, 1906, p. 39, the sixteenth century records the struggle of the word million for its place in numeration.)

In Lambeck's (No. 55, p. 17.) arithmetic (c. 1661) we read:

Nach der neuen und bequemen Art zu numeriren, da man tausend mal tausend, eine Million; tausend mal tausend Millionen, eine Billion; tausend mal tausend Billionen, ein Trillion, nennet etc.; stehet die vorgegebene Zahl also eingetheilet 41,234567,894.

In Italy millioni or milione, a variation in spelling, occurs in Ghaligai (1552), Peverone (1558), Unicornio (1598), and in the seventeenth century in Griminelli, Venturoli, Bonvicino, Bassi, and Ciacchi. The French writers Chauvet, Prestet, Monier, and Blondel give only the form Millions, while the works of Aurel and Puig give the Spanish cuentos. (Cantor, Vorlesungen über Geschichte der Mathematik, v. 2, p. 387, says that Ciruelo (1505) called 10^6 cuentos, and 10^{12} millon, according to the Spanish custom.)

For 10^9 Chauvet (No. 21, f. 2, r.) uses the multiplicative principle when he writes mil millions; Prestet (No. 70, p. 7.) has the special form milliars, and Lucas (No. 58, p. 3.) uses milleni. We have in Blondel (No. 9, p. 52.) an instance of the use of bimillion for 10^9 , and trimillion for 10^{12} , which is still customary in France today. (See Tropicke, v. 1, 3d. ed., p. 9.) It has been noted that Lambeck (No. 55, p. 17.) mentions 10^{12} read as Billion according to the newer and easier method; Lucas (No. 38, p. 3.) also uses biliones in preference to the older forms.

In the translation of the numeral word for zero from the Arabic as-sifr into the Latin, we have zephirum (Leonard of Pisa), cifra (Nemorarius), and their many forms. Although the word chiffre gradually assumed a new meaning in France in the fourteenth century, we still find an instance of its use as a numeral word for zero in 1627. In Chauvet (No. 21, ff. 1, v., 3, r.) we read: "le dixiesme s'appelle vulgairement chiffre, les uns l'appellent zero, comme les Arabes." Chauvet actually uses chiffre for zero.

The use of siphram by Tunstall (1538), cyphram by Gemma (1549) and cypher by Moore has been noted above. Noviomagus (No. 65, f. 22, r.) writes sipheram or zyphram, which latter form is found also in Regius (No. 76, f. 49, r.). In Reisch (No. 77.) and in Micyllus (No. 60, p. 2.) the cifra of Nemorarius is mentioned alone. Napier (No. 65, p. 5.) gives cifra or zero, whereas Wolphius (1534), Peverone (No. 68, p. 6.), and Forestani (No. 35.) give cifra (zifra) or nulla respectively. Nulla was used exclusively by Rudolff (No. 81, f. 2, v.), Felner (No. 30, f. 3, v.), Suevus (No. 88, p. 3.), and, as noted before, by Jacob (1612). Among those who give nulla and zero, as did Feliciano (1550), are Bonvicino (No. 12, p. 1.), and Bassi (No. 6, p. 4.). Together with these words we find Bonvicino using the peculiar term Nota.

Graffenried (No. 42.) and Curtius (No. 27.) give no alternate forms for nulla. However, Lambeck (No. 55, p. 16.)

(c. 1661) uses Null for the tenth 'Ziffer', and underneath it writes Nichts, thus:

Null

o

Nichts, in the century before the writers mentioned by Tropicke as using the term null or nulla. (See Tropicke, v. 1, 3d ed., p. 12.) An equivalent term for Nichts is that of nihil listed in Cardan (No. 16, f. 5, r.), but spoken of as nullus in the running text. Steinmetz (No. 86, ff. 8, v., 6, r.) gives cyphram or nihili but uses the former. Ramus in both the 1599 and 1613 editions says circulus for 0, which form was common in the early treatises before the introduction of printing. (See Susan R. Benedict, A Comparative Study of the Early Treatises Introducing into Europe the Hindu Art of Reckoning, p. 32.).

We have mentioned the change from nulle or tziphra to zero in the 1587 and 1619 editions of Finé and Savonne respectively. Clavius (No. 24, p. 8; No. 25, p. 11.) writes cifra in 1585, but in the following year and in his seventeenth-century editions he says cifra o zero.

Zero was used exclusively, as noted above, by Cataneo (1546), and also by the Spanish writers Aurel (No. 3, f. 1, v.) and Puig (No. 72, p. 60.), by the Italians Manelli (No. 59, p. 2.) and Venturoli (No. 94, p. 3.) and the French Prestet (No. 70, p. 6.), Monier (No. 62, p. 5.), and Blondel (No. 9, p. 50.). The latter writes:

....il ne sert qu' à remplir les sieges ou places
entre les chiffres, qui demeurant vuides pourroient
causer de la confusion.

The German Waser (No. 99, f. 10, r.) in his Roman notations, gives five word forms for zero, the Hebrew en, Arabic Siphra, German Ziffer, Greek θδω, the Latin Nulla or Nullum; but he himself resorts to a sixth word, circulus. Alsted (No. 2, p. 24.) lists nulla, ciphra, and circulus, but uses ciphra. Blasius (No. 8.) calls 0 theta, circulus, and cifra, as does Husswirt. Both Unicornio (No. 93, p. 3.) and Griminelli (No. 43, p. 2.) give nulla, cifra (ziffra) and zero, but the former uses commonly nulla, and the latter, zero.

B. FUNDAMENTAL OPERATIONS

1. Fundamental Operations in General

In his translation of Rahn's Algebra in 1668 Thomas Brancker (No. 73, p. 2.) states:

There are in Arithmetick six distinct single operations, all of which are performed in Algebra, viz., Addition, Subtraction, Multiplication, Division, Involution, Evolution;...

The German Felner (No. 30, ff. 5, r.-9, r.) speaks of 'die Species' as 'Addiren, Subtrahiren, Duplirn, Medirn, Multiplirn, Dividirn.' Scheubel (No. 83, pp. 71-78.), in 1560, like Brancker, lists six species, but he substitutes progressions for involution.

Followers of Sacrobosco (See Susan R. Benedict's thesis on the early treatises) in teaching the following nine fundamental operations are Husswirt (1501), Micyllus (No. 60, p. 2.), and Steinmetz (No. 86, f. 7, v.): numeration, addition, subtraction, duplation, mediation, multiplication, division, progressions, roots. There are others in the sixteenth century who give seven, omitting duplation and mediation, as Cardan (No. 16, f. 6, r., who thus early speaks of them as operations rather than species), Regius (No. 76, f. 50, v.), Noviomagus (No. 65, f. 8, r.), and Unicornio (No. 93, ff. 4, v., 5, r.). (See D. E. Smith, Source Book in Mathematics, p. 18, for Recorde's account of the seven operations.). Still others, like Tagliente as early as 1520, and Vuelpius (No. 98, f. 5, r.), teach only five, excluding progressions and roots. Aurel (No. 3, f. 4, r.) speaks of the same five Species, but he calls addition, subtraction, multiplication, and division the 'quarto reglas principales.' By the end of the seventeenth century the number of fundamental operations had quite generally been reduced to at most these five: numeration, addition, subtraction, multiplication and division, as in Puig (No. 72, p. 578.) and Lucas (No. 58, p. 1.). Tartaglia (No. 90, f. 3.) discusses the question of the number of fundamental operations at some length:

Le specie, . . . sono noue, . . . Numeratione, ouer Representatione, . . . Additione, (cioe Sommare, ouer Raccogliere) . . . Sottratione (cioe Sottrare, ouer abattere, ouer Cavare, ouer Restare) . . . Dupplatione, (cioe doppiare) . . . Multiplicatione, (cioe Moltiplicare) . . . Mediatione (cioe dimezzare, . . .) . . . divisione, (cioe Dividere, ouer Partire) . . . Progressione, . . . è detta estratione de Radice, Et queste noue specie alcuni le chiamano, Atti, altri gli dicono Passioni del Numero, & perche lo indoppiare non si distingue dal Moltiplicare, ne lo Dimezzare dal partire, molti hanno detto, & determinato, Le sopradette noue specie Atti, ouer Passioni esser solamente sette, cioe Numerare, Sommare, Sottrare, Moltiplicare, Partire, Progressioni, & estration, ouer cauar de Radice, Ma perche le Progressioni, & l'estration de Radice, non sono necessarie in alcun caso, realmente accadente in tutte l'arte Negotiararia, della quale solamente quiui trattar intendo. . .

In the second half of the sixteenth century there are those who reduce the number of operations to four, omitting numeration. These include Cataneo (1546), as noted above, Feliciano (1550), Finé (1555), Sfortunati (1561), and Savonne (1571). As early as 1515 Ortega treats the four species with integers and fractions. (However, D. E. Smith in his History of Mathematics, v. II, p. 33, gives Gemma Frisius the credit of being one of the first of the writers of any note to reduce the number to four.). There are these representative works during the seventeenth century which give four operations: Waser (1603), Chauvet (1627), Lambeck (c. 1661), Griminelli (1670), Prestet (1675), Monier (1693), and Blondel (1699). The first three speak of them as 'species'; Griminelli (No. 43, p. 1.) says that arithmetic is divided 'in quattro parti, o specie'; Prestet (No. 70, pp. 10, 19, 25.) uses the expressions: 'De l'Addition simples', 'De la Soustraction simples', 'De l'Addition composée ou Multiplication', 'De la soustraction composée, ou Division.' On the other hand, by the terms 'Addition and Soustraction composée' Monier (1693) refers to addition and subtraction of denominate numbers. Blondel (No. 9, pp. 60, 61) says:

il ya quatre Operations de l'Arithmetique pratique que l'on appelle les quatre premieres regles. . . Ces quatre regles sont le fondement non seulement de tout ce que se peut pratiquer en Arithmetique, mais mêm des principales operations du reste des Mathematiques;

. . .

Duplation and mediation are among the fundamental operations, as noted above, in the works of Husswirt (1501), Wolphius (1534), and Peurbach (1539). Husswirt has duplation and mediation follow multiplication and division respectively, whereas Wolphius has them precede. Felner (1535), Peurbach (1539), Micyllus (1555), and Steinmetz (1575) place both duplation and mediation before multiplication and division, and Husswirt does the same in counter reckoning. Rudolff (No. 8, f. 35.) lists doubling and halving only in counter reckoning, while Scheubel (No. 83, p. 112.) gives them only for fractions; Tunstall (1538), Suevus (No. 88, p. 2.), and Alsted (No. 2, p. 27.) merely mention these operations as included by some among the fundamental ones. By the end of the sixteenth century these two operations had largely fallen into disuse; an exception noted above, as late as 1660, is Moore, who still speaks of 'bypartition and duplication.' (D. E. Smith, History of Mathematics, v. II, p. 35, says that these processes are rarely found in printed arithmetics of Italy, Spain, France, or England.).

The order of the four fundamental operations advocated by Köbel (1515), as addition, subtraction, multiplication, and division, was generally adopted by the writers of the sixteenth and seventeenth centuries. Calandri (1518), however, and Tagliente (1520), as noted above, followed the order of Borghi (1484); namely, multiplication, division, addition, and subtraction.

The use of symbols for these fundamental operations during the two centuries in question has been discussed at some length above, and it has been shown that the symbols + and - were in abundant use in the seventeenth century. The references to such popular and influential works as those of Vieta (1630), Descartes (1649), Moore (1660), Renaldini (1665), Feliciano (1669, 1692), Prestet (1675), Hemeling (1678), and Blondel (1699) are quite essential in the discussion of this topic. These writers are not mentioned by Tropicke (v. I, 3d. ed., p. 81.), who ascribes incorrectly the spread of these symbols among the common

people to the German writers of the eighteenth century.

In regard to checks, Rudolff (1526) uses 6, 7, and 8; however, in 1532 he checks largely by 9, mentioning also the checks by 7 and 11. He adds (No. 8, f. 12, v.):

Die gewissest prob / so man gehalten mag / ist / wañ
ein Species die ander probirt / wie nachvolgt,

indicating that the contrary operation is the sure test. This last method of checking is used exclusively by Felner (1535) and Waser (1603). The majority of the writers of these centuries check by 9 alone, as do Aurel (1552), Micyllus (1555), Suevus (1595), in addition to Tagliente (1520), Gemma (1549), and Moore (1660), mentioned above. Many check by the more common way, by 9 and 7, as Cardan (1539), Regius (1543), Tartaglia (1556), Bassi (1666), Griminelli (1670), Husswirt (1501), Cataneo (1540), Feliciano (1550), Sfortunati (1561), and Lapazzaia (1601). Wolphius (1534) has been noted as giving the checks with 7 only, while Jacob (1612) mentions the check by 9, 3, 7, 8, 11, etc. A further peculiarity is introduced by Steinmetz (No. 86, f. 18, r.), who uses the check by 9 but gives a table for checks by 7, 11, and 13. Tartaglia (No. 90, ff. 8-12.) gives a lengthy discussion of checks, especially by 9 and 7.

2. Fundamental Operations in Particular

a. Addition

In the treatment of tables, Tropfke (v. I, 3d. ed., pp. 140-148) omits reference to the addition tables found not only in Tartaglia (1556), as noted above, but also in Tunstall (1538), Peverone (No. 68, p. 7.), and in Hemeling as late as 1678. (L.L. Jackson, Sixteenth Century Arithmetic, p. 41, says that the writers who used addition tables were the exception, and he cites Tartaglia as the exception. F. Unger, Die Methodik der Praktischen Arithmetik, Leipzig, 1888, p. 73, refers to the use of addition tables up to $90 + 90$ by Leonard of Pisa.)

Noviomagus (No. 65, f. 23, v.), Chauvet (No. 21, f. 3, v.), Puig (No. 72, p. 62.), Prestet (No. 70, p. 13.), Hemeling (1678), and Monier (No. 62, p. 10.) teach addition downwards. Lambeck (No. 55, p. 18.), however, instructs

the reckoner:

Fange unten bei der rechten Hand an und gebrauche
das Wort: Und,

writing the word und between each of the corresponding digits of two summands. A survival of the Hindu method of adding from left to right is seen in Ciacchi (No. 22, p. 61.), who uses this procedure as one of his checks for addition.

Monier (No. 62, p. 10.) writes the numbers to be carried below the digits of the sum and scratches them out after adding them, thus:

$$\begin{array}{r} 3456 \\ 5643 \\ 4652 \\ \underline{7866} \\ 21617 \\ \hline 221 \end{array}$$

Regius (No. 76, f. 54, f.) illustrates addition of only two numbers, and scratches out the digits as added in this manner:

$$\begin{array}{r} 642 \\ \underline{334} \\ 976 \end{array}$$

This process is connected with the old scratch method taught before the introduction of printing. Save for such variations, the addition problems of these early texts appear in the same form as they do today.

b. Subtraction

Three methods of subtraction may be found simultaneously in the arithmetics of these two centuries; namely, the two borrowing methods and the so-called complementary method.

Tunstall (1538) illustrates the first borrowing method very clearly in the following manner:

$$\begin{array}{r} 2 \quad 9 \quad 10 \quad 10 \\ \underline{1 \quad 1 \quad 1 \quad 1} \\ 3 \quad 0 \quad 1 \quad 0 \\ \underline{1 \quad 1 \quad 1 \quad 1} \\ 1 \quad 8 \quad 9 \quad 9 \end{array}$$

In addition to the eight writers of the sixteenth century mentioned by Tropfke (v. I, 3d. ed., p. 90.) as using this

first method, we wish to mention eight others of importance: Noviomagus (No. 65, f. 25, r.), Ghaligai (No. 38, f. 4, v.), Peverone (No. 68, p. 11.) Finé (1587), Clavius (No. 25, p. 19.), Bonvicino (No. 12, p. 6.), Bassi (No. 6, p. 14.), and Prestet (No. 70, p. 15.). Bassi (No. 6, p. 14.) says that this method was used 'dalli nostri Antichi, benché alcuni ancora al presente se ne servono.'

Hemeling (1667, 1678) performed subtraction by a second method, that is, the addition of one to the next highest number in the subtrahend, introduced into Europe by Leonard of Pisa and one of the most common methods of subtraction in the sixteenth century. (See L. L. Jackson, The Educational Significance of Sixteenth Century Arithmetic, New York, 1906, p. 51.). Followers of this method in the seventeenth century are Manelli (No. 59, p. 10.), Bonvicino (No. 12, p. 6.), Bassi (No. 6, p. 14.), Griminelli (No. 43, p. 29.), Prestet (No. 70, p. 16.), and Monier (No. 62, p. 30.).

These two methods are compared by Bonvicino in the subtraction problem

$$\begin{array}{r} 84065 \\ 68478 \\ \hline 15587, \end{array}$$

when he says:

Altri in vece di aggiungere la decina al numero susseguente la levano realmente dal numero di sopra, coma sarebbe levato il primo 8. di 15. poiche la decina fù levata dal 6. nel secondo numero, in vece di dire 8. dal 16. dicono 7. dal 15. mà è una cosa istessa, e perche genera qualche confusione, ritengo la prima via per più facile.

The additive method of subtraction is also suggested by him in the following words:

Altri in vece di dire 8. di 15. reman 7. dicono 8. ad andare al 15. vene vogliono 7. poi operano come è detto, mà tutto è una cosa istessa. Può adoprarsi qual modo à cadauno riesce più facile, che nul la rileva.

Complementary subtraction, in which the difference of the digit in the subtrahend from ten is added to the respective minuend digit, was very popular during these centuries. It appeared, as noted above, in the works of Gemma Frisius (1549), Finé (1555, 1587), Savonne (1571), Lapazzaia (1601),

and Jacob (1612). Tropicke (v. I, 3d. ed., p. 94.) has mentioned the earlier works of Finé (1535) and Jacob (1565). Besides the nine authors of the sixteenth century listed by Tropicke we find the complementary method used by Reisch (No. 77.), Felner (No. 30, f. 10, v.), Steinmetz (No. 86, f. 20, v.), Unicornio (No. 93, p. 12.), and in the seventeenth century by Clavius (No. 25, p. 19.), Alsted (No. 2, p. 31.), Bassi (No. 6, p. 14.), Puig (No. 72, p. 65.), Lucas (No. 58, p. 8.), and Blondel (No. 9, p. 72.). Clavius (No. 25, 1618, p. 20.) calls this method 'Più facil regola di sottrarre', whereas he states that the first borrowing method 'è usata da molti Aritmetici.' Likewise Tartaglia (No. 90, f. 15, v.), to whom Tropicke has referred (v. I, 3d. ed., p. 94.), makes this statement with reference to this method:

. . . il terzo dappoi che si ha inteso è alquãto piu leggiadro & accommo da maneggiare da cadauno de gli altri duoi. . .;

and Bassi says:

. . . e questo terzo modo quasi da tutti viene usato, per essere grandemente facile.

Tropicke has noted (Ibid., p. 92.) that Ramus in his arithmetic of 1592 begins his subtraction on the left, according to the Hindu custom. We find this true also in the 1599 edition. Concerning this order of subtraction we quote from Prestet (No. 70, p. 17.):

Cette methode me semble plus d'usage, & je m'en sers plus volontiers que des precedentes, parceque je trouve qu'elle charge beaucoup moins la memoire que celle du probleme, & qu'outre cela il ne faut effacer ni ajouter aucuns chiffres, ce qui est fort commode pour les supputations longues & frequentes, où le meilleur est toujours d'embarasser ses nombres & ses calculs le moins qu'on le peut.

Alsted (No. 2, p. 31) gives directions that the remainder may be written above or below. Chauvet (No. 21, f. 13, r.) makes his subtraction clear in this manner:

$$\begin{array}{r} 799 \\ \cancel{8000} \\ \underline{3459} \\ 4541 \end{array}$$

Various symbolic aids in the subtraction of a larger number from a smaller number are introduced in the early books. We find a dot placed above or to the right of the numbers in the minuend to indicate a borrowing in the complementary method, as by Blondel (No. 9, p. 72.) and Savonne (1571) respectively. Alsted (No. 2, p. 31.), Felner (No. 30, f. 10, v.), Graffenried (No. 42.), and Lucas (No. 58, p. 8.) place the dot above, to the right, or below the numbers in the subtrahend respectively, to denote the addition of one in this same method. In other respects the subtraction problems resemble those of today.

c. Multiplication

Italian writers of these centuries, following Pacioli, include commonly various methods of multiplication in their texts. This was the case with Tagliente (1520), Feliciano (1550), and as late as 1678 with Figatelli. Unicornio (1598), Manelli (1659), and Bassi (1666) display many methods, among them our modern method called 'per Baricoccolo', the so-called lightning method 'per Crocetta', the chess-board method 'per Quadrato', and 'per modo allo Indietro' or 'alla Roverscia.' Besides these we find in texts of this time the methods 'per Ripiego', 'per Scapezzo', complementary multiplication, 'per Triangolo', 'per Piramide', and others.

Our method, 'per Baricoccolo', is very popular and may be found among other methods in practically all of the works of this period, with the important exception of Chaligai, (1552) who seems to have a preference for multiplication 'allo Indietro.' Felner (No. 30, f. 13, r.), Aurel (No. 3, f. 6, r.), Girard (No. 39, f. 6, r.), and Prestet (No. 70, p. 21.) follow Cardan (1539) in teaching only our method.

Multiplication 'per Ripiego', in which the multiplicand is multiplied by one factor of the multiplier and this result by the other factor, is included by Steinmetz (No. 86, f. 33, v.), Unicornio (No. 93, ff. 19, v., 21, v.), Puig (No. 72, p. 71.), and Venturoli (No. 94, p. 4.). The 'assai ammaestreuole et ingenioso' (No. 93, f. 19, v.) method 'per Crocetta' is given by Cantone (No. 15, p. 38.), Steinmetz (No. 86, f. 35, r.), and Venturoli (No. 94, p. 5.). A way of multiplication 'molto bella & facile', multiplication 'per Quadrata', is displayed by Peverone (No. 68, p. 14.); Regius (No. 76, f. 64, r.) calls this 'modus elegans.'

In multiplication 'allo Indietro', the partial products are obtained by beginning with the highest number of the multiplier. Ghaligai (No. 38, f. 6, r.) multiplies 987 by 876 thus:

$$\begin{array}{r}
 876 - 978 \\
 \hline
 782400 \\
 68460 \\
 5868 \\
 856728
 \end{array}$$

This method may be explained in this manner:

$978 \times 876 = 978 (800 + 70 + 6) = 782400 + 68460 + 5868 = 856728$. This problem and this form are almost exactly given by Juan Diez Freyle in Mexico in 1556 in the first arithmetical work printed in the New World. Napier (No. 64, p. 39.) and Ciacchi (No. 22, p. 17.) use this method also, but, like Hemeling, omit the zeros. The method 'per Scapezzo' is similar in procedure, in that the multiplicand is multiplied in turn by the digits whose sum is equal to the multiplier, and the results thus obtained are added. We find this example, 34×16 , in Unicornio:

$$\begin{array}{r}
 34 \quad 34 \quad 34 \quad 34 \\
 \underline{16} \quad \underline{3} \quad \underline{4} \quad \underline{9} \\
 544 \quad 102 \quad 136 \quad 306 \\
 \quad \quad \quad \underline{136} \\
 \quad \quad \quad \underline{102} \\
 \quad \quad \quad 544
 \end{array}$$

This same process is used by Alsted (No. 2, p. 34.) in multiplying 8 by 7 in this fashion:

$$\begin{array}{r|l}
 8 & \\
 2. & 3. \quad 2. \\
 \hline
 16. & 24. \quad 16.
 \end{array} \quad 7$$

56.

In the texts examined, the complementary method is based upon these formulas, both of which may be found in treatises of the thirteenth century (See Susan R. Benedict, A Comparative Study of the Early Treatises Introducing into Europe the Hindu Art of Reckoning, pp. 82, 88, 89, 99, 100, 103):

$$1. a.b = 10a - a(10 - b)$$

$$2. a.b = 10[a - (10 - b)] + (10 - a)(10 - b).$$

Not only Regius (1543), as noted above, uses the first formula, but also Scheubel (No. 83, p. 37.). Both formulas are used by Rudolff (No. 81, ff. 5, r., 6, r.) in his work of 1532. (Tropfke, v. 1, 3d. ed., p. 101, refers to the 1526 edition.). Formula two is more frequently employed in the majority of the works of this time, as in Husswirt (1501), Gemma (1549), Finé (1555), Micyllus (1555), Steinmetz (1575), and Blondel (1699), mentioned previously, also in Vuelpius (No. 98, f. 15, v.), Noviomagus (No. 65, f. 25, v.), Clavius (No. 25, p. 28.), Graffenried (No. 42, pp. 21, 22.), and Monier (No. 62, p. 55.). A similar principle is used by Monier (Ibid., p. 56.) in the 'methode par les doigts', and before him by Moore (1660).

A sort of 'filling-in' process, as used by Steinmetz (No. 86.) and Behme (No. 7, p. 38.), is illustrated by the following example (No. 86, f. 31, r.):

$$\begin{array}{r} 365 \\ 24 \\ \hline 1220 \\ 124 \\ 610 \\ 2 \\ \hline 8760 \end{array}$$

Lucas has a chapter entitled 'Arithmetica Rabdologica', which teaches the use of Napier's rods for multiplication. The use of 'Nepairs bones', as Napier's rods are vulgarly called, was also taught and recommended in the arithmetics of Jonas Moore (1660) and William Leybourn (1660, p. 265.).

An instance of actual multiplication with zero is found in Vuelpius (No. 98, f. 16, r.) who gives this note:

Figura significativa in ziphram, sive circulum ducta,
ziphram producit: ut,

$$\begin{array}{ccc} 0 & 0 & 0 \\ \underline{2} & \underline{3} & \underline{4} \\ 0. & 0. & 0. \end{array} \begin{array}{l} \text{multiplicandae.} \\ \text{multiplicantes.} \\ \text{productae.} \end{array}$$

Siphra, sive circulus in figuram significativam
ductus, ziphram gignit,

$$\begin{array}{ccc} 1. & 2. & 3. \\ \underline{0.} & \underline{0.} & \underline{0.} \\ 0. & 0. & 0. \end{array} \begin{array}{l} \text{multiplicandi} \\ \text{multiplicates} \\ \text{productae.} \end{array}$$

Ziphra, sive circulus in zipfram ductus, ziphrā procreat.

$$\begin{array}{r} 0. \quad 0. \quad 0 \quad \text{multiplicande} \\ 0. \quad 0. \quad 0 \quad \text{multiplicantes} \\ \hline 0. \quad 0. \quad 0. \quad \text{productae.} \end{array}$$

It may be of interest to add here that zero combinations are recently being emphasized in instruction in arithmetic.

d. Division

It may be of interest here to quote Rahn's definition of division (No. 73, p. 2.):

Division is to take the same quantity from another as oft as may be, as

$$\begin{array}{r} 6 \quad 4 \quad 2 \\ 2 \quad 2 \quad 2 \quad | \quad 3 \quad . \\ \hline 4 \quad 2 \quad 0 \end{array}$$

Remnants of a Hindu method of division, consisting of decomposition of the divisor into factors, may be found in Tartaglia (No. 90, f. 25, v.), Unicornio (No. 93, f. 26, v.), Manelli (No. 59, p. 18.), and Venturoli (No. 94, p. 7.). Tartaglia's explanation of this division 'per Repiego' may be seen in the problem,

$$5867 \div 48 = 122 \frac{11}{48} \quad \text{where the chosen factors are 8 and 6:}$$

$$\begin{array}{r} 8 \overline{) 5867} \\ 6 \overline{) 733} - 3 \\ 122 - 1 \end{array}$$

To obtain the final remainder add the product of the first divisor and the last remainder to the first remainder, in this case, $8 \cdot 1 + 3 = 11$.

Among other methods of division, division 'per Galera' is rarely missing in the arithmetics of these centuries. In the sixteenth century, Rudolff (No. 81, f. 10, r.), Felner (No. 30, f. 15, r.), Regius (No. 76, f. 68, r.), Noviomagus (No. 65, f. 30, r.), Aurel (No. 3, f. 8, v.), Micyllus (No. 60, p. 29.), Peverone (No. 68, p. 17.), Scheubel (No. 83, p. 49.), and Steinmetz (No. 86, f. 41, r.) teach only this method. In the seventeenth century, Gautruche

(No. 36, p. 10.) (a work used in Harvard College), and Blondel (No. 9, p. 110.) still cling to it. On the other hand, a few of the later writers like Bonvicino (1665), Venturoli (1681), and Lucas (1697) do not advocate it. In fact, Bonvicino remarks (No. 12, p. 10.):

Lasciarò da parte il partire per Galea, perche' è
oggi poco usato, come quello, che per la molteplicità
de numeri, e per il molto scassamento, genera confusione,
e darò il modo di dividere, per la danda, prima lunga,
poi corta.

Division 'per Batello', according to Manelli (No. 59, p. 22.), is similar to division 'per Galera', except that the numbers in the process are written below the dividend instead of above. Manelli illustrates as follows:

$$\begin{array}{r} 451 \overline{) 94746} \mid 210 \\ 145 \\ 0036 \\ 003 \end{array}$$

We must not confuse the latter method with 'per Danda corta', or 'mezza Danda', taught in many Italian works of the seventeenth century, as in Clavius (No. 25, p. 59.), Napier (No. 64, p. 54.), Manelli (No. 59, p. 20.), Bonvicino (No. 12, p. 15.), Bassi (No. 6, p. 44.), Griminelli (No. 43, p. 64.), and Venturoli (No. 94, p. 7.). In both, the work is carried out beneath the dividend and the partial products are not written; but in division 'per Danda corta', the multiplications and subtractions are performed from right to left, and we may see this from the following problem in Manelli:

$$\begin{array}{r} 451 \overline{) 94746} \mid 210 \\ 0454 \\ 0036 \end{array}$$

(For the methods used by Clavius, see L. G. Simons, "Historical Notes on Arithmetical Division", Scripta Mathematica, June, 1933, pp. 362, 363.).

In 'per Danda longa', our present-day method, the partial products are recorded. It is used particularly in the works of these Italian and Spanish writers who offer so many divers methods of operation: Unicornio (No. 93, f. 34, r.), Forestani (No. 35, ff. 16-25.), Manelli (No. 59, p. 21.),

Bassi (No. 6, p. 49.), Griminelli (No. 43, p. 68.), Puig (No. 72, p. 81.), and Venturoli (No. 94, p. 7.). Manelli, in comparing the two most used methods, 'per Danda', and 'per Galera', says (No. 59, p. 19.):

. . . e questo li pare più facile il partire à Danda, per li principāti è più sicuro, perche si vede il moltiplicare è il sottraere; mà il partire à galera è più corto, ma fà di bisogno star bene avvertito nel tagliar le figure, perche errando, non si può si facilmente correggere, come la persona non è ben pratica.

Prestet (No. 70, p. 31.) also calls our method 'un peu plus longue à la vérité, mais plus facile à observer.'

In Prestet (*Ibid.*), also, we find an instance of the repetition of the divisor below each remainder, similar to the practice of Karsten (1767). (See Tropfke, v. I, 3d. ed., p. 109.). There are still a few, like Coutereels (No. 26, p. 30.), Griminelli (No. 43, p. 68.), and Behme (No. 7, p. 37.), who follow Finé as noted above, in writing down in advance the partial products of the divisor. It should be noted further that Ghaligai does not treat division of integers, but speaks of division 'per Ripiego', 'per Danda', and 'per Galera', only in connection with denominate numbers.

C. FRACTIONS

1. Common Fractions

It is usual to find a fraction defined as 'pars integri.' Cardan (No. 16, f. 7, r.), however, says:

Unde nihil aliud est dicere $\frac{3}{5}$ quam .3. divisum per .5.

Prestet (No. 70, p. 42.), too, regards it as a division or ratio. A little later, Monier (No. 62, p. 142.) remarks:

Les fractions ne sont que l'expression de la raison que est entre l'entier & sa partie.

Girard (No. 39, p. 8.) goes further and defines it as 'une division imparfaite.'

Fractions in general are still classified as vulgar or common, and astronomical or sexagesimal by Finé (1555), Regius (No. 76, f. 83, v.) and Bassi (No. 6, p. 70.). Ordinarily writers of these centuries subdivide common fractions into simple fractions, and fractions of fractions, as do Regius (No. 76, f. 83, v.) and Aurel (No. 3, f. 10, v.). Lambeck (No. 55, p. 72.), however, says that fractions of fractions belong to multiplication. Rudolff (No. 81, f. 18, v.) and Felner (No. 30, f. 27, v.) call them 'schlechte brüch' and 'brüch von brüchen', whereas Vuelpius (No. 98, f. 29, v.) and Lucas (No. 58, p. 55.) use the terms 'primaria' and 'secundaria.' Girard (No. 39, p. 8.) speaks of proper and improper fractions, and in the latter he includes such fractions as $5/4$, $6/2$, and even $6/1$. Lambeck (No. 55, p. 72.) says that in trade there occur

'Gemengte Brüche' as $\frac{2\frac{1}{2}}{4}$, and $\frac{3\frac{1}{2}}{4}$, which he reads as

'Drittehalb Viertheil' and 'Viertehalb Viertheil.'

The majority of writers mention the value of a fraction according as the numerator is greater than, equal to, or less than the denominator. Some, like Cardan (No. 16, f. 21, v.), Alsted (No. 2, p. 45.), Bonvicino (No. 6, pp. 18, 19.), Bassi (No. 6, p. 55.), and Lucas (No. 58, p. 56.), observe that a fraction decreases in value if, keeping the same numerator, the denominator is increased, and on the contrary, increases if, keeping the same denominator, the numerator is increased.

In listing the operations for fractions, most authors treat numeration, reduction, and the four fundamental operations. Some, like Felner (No. 30, f. 27, v.) and Scheubel (No. 83, p. 91.), include doubling and halving, and a few, like Cardan (No. 16, 21, v.) and Scheubel (No. 83, p. 91.), add roots and progression. The notation used in writing fractions has been discussed previously. Scheubel (Ibid.) and Alsted (No. 2, p. 44.) give very definite directions for the reading of a fraction in saying that the numerator is read as a cardinal number, the denominator as an ordinal numeral.

Reduction of fractions to higher or lower denominations depends upon the property that multiplying or dividing both terms of a fraction by the same number does not alter the value of the fraction. This is clearly enunciated by Lucas

(No. 58, p. 57.). According to Bassi (No. 6, p. 56.), Gautruche (No. 36, p. 13.), and Blondel (No. 9, p. 150.), those fractions are defined as equivalent whose numerators have the same ratio to their denominators. In order to find the greater of two fractions recourse is usually had to reduction to a common denominator, but Steinmetz (No. 86, f. 65, v.), in determining which is the greater, $3/4$ or $1/2$, does what is equivalent to reduction to decimals, thus:

$$\begin{array}{r} 30 \overline{) 7} \qquad 10 \overline{) 5} \\ 4 \qquad \qquad 2 \end{array}$$

Girard calls fractions reduced to their lowest terms, primitive, and the fractions themselves, derivative, as $1/2$ and $5/10$ respectively.

The process of obtaining the primitive fraction by finding the greatest common divisor, called by Lucas (No. 58, p. 57.) 'aureum numerum', is usually performed by the method of Euclid, Book 7, Prop. 2. This may be illustrated briefly from the explanation given by Clavius (No. 25, p. 79.)

in the reduction of $\frac{60}{96}$:

$$\begin{array}{r} 96 \div 60, \text{ rem. } 36 \\ 60 \div 36, \text{ rem. } 24 \\ 36 \div 24, \text{ rem. } 12 \\ 24 \div 12, \text{ rem. } 0 \\ \text{G.C.D.} = 12 \end{array}$$

There are others who find the common divisor by trial, and a few, like Jacob (1612) mentioned previously, present tests for divisibility by 2, 3, 4, etc. Rudolff (No. 81, f. 22, r.) gives the familiar tests for divisibility by 2 and 5; and his test for 3 is that 0, 3, or 6 remains after the check by 9. Lucas (No. 58, pp. 57, 58.) gives the usual tests for 2, 5, and 10, to which Blondel (No. 9, p. 153.) adds these rules, some of which are rarely used:

Tout nombre dont les caracteres ajoutés ensemble font 3 ou un multiple de 3, peut être divisé par 6, si le dernier chiffre est nombre pair.

Et par 15, si le dernier caractere est 5.

Et par 30, si le dernier caractere est 0.

Tout nombre dont tous les chiffres ajoutés ensemble font 9 ou un multiple de 9, peut être divisé par 18 si le dernier caractere est nombre pair.

Et par 45, si le dernier chiffre est 5.

Et par 90, si c'est un 0.

There is nothing extraordinary in the treatment of addition and subtraction of fractions in the textbooks of these centuries. Rudolff (No. 81, f. 20, v.) treats multiplication before the other three operations and he explains it simultaneously with 'Teil suchen.' We may add the writers Bonvicino (No. 12, p. 23.) and Blondel (No. 9, pp. 171, 172.) to those given by Tropicke as enunciating the fact that the product of two fractions is less than either. As regards division of fractions we quote from Bonvicino (No. 12, p. 24.):

Terciò ogni qual volta la minuzia, che divide è minore di quella, che deve esser divisa, il quoziente farà maggiore, perche tante più volte v'entra: come all'incontro, quando il divisore è più della frazione, che deve esser divisa, il quoziente è minore, e quanto è maggiore, tanto minore farà il quoziente, perche tante meno volte v'entra.

Multiplication is usually performed with the aid of parallel lines, and division by means of crossed guiding lines. In dividing $\frac{3}{4}$ by $\frac{2}{3}$, many writers, like Cardan (No. 16, f. 21, v.), Peverone (No. 68, p. 28.), Unicornio (No. 93, f. 46, v.), Manelli (No. 59, p. 33.), and Venturoli (No. 94, p. 16.), place the divisor on the left in an arrangement similar to this: $\frac{2}{3} \times \frac{3}{4} \quad \frac{9}{8}$. Some place the divisor on the right, thus: $\frac{3}{4} \times \frac{2}{3} \quad \frac{9}{8}$. We may mention many other writers besides Clavius who make use of the principle of inversion: Waser (No. 99, f. 40, v.), Graffenried (No. 42, p. 112.), Alsted (No. 2, p. 49.), Lambeck (No. 55, p. 86.), Bonvicino (No. 12, p. 24.), Bassi (No. 6, p. 62.), Venturoli (No. 94, pp. 15, 16.), and Lucas (No. 58, p. 78.). Alsted (No. 2, p. 49.) remarks:

Facilius hoc praestiteris, si divisore inverso procedas ut in multiplicatione.

Again Bonvicino (No. 12, p. 23.) says:

Convieni scrivere il partitore al reverscio di quello si scrivono li rotti.

Lucas (No. 58, p. 78.) directs briefly as follows: "In praxi invertitur dividens." The method of inversion is rarely used in the sixteenth century (See L. L. Jackson, Sixteenth Century Arithmetic, p. 107.).

2. Decimal Fractions

The history of decimal fractions in the sixteenth century has been completely treated by Tropfke. In this connection, Oughtred's Clavis Mathematicae is cited by him (v. I, 3d. ed., p. 181.). Another important reference for this subject is to the arithmetic of Sir Jonas Moore (No. 63.), who was a close follower of Oughtred. In his development of decimal fractions, Moore refers several times to 'The Learned Mr. Oughtred in his Clavis Limata.' (Ibid., p. 14 in preface.).

In speaking of fractions, Moore states (Ibid., p. 6.):

. . . we use a more briefe expression of some kindes of them by omitting the Denominators, when the parts are commonly known, . . .

As because the twentieth parts of a pound are famously known by the name of shillings, we use to say and write 2s. 3s. . . briefly with the Numerators onely, and not fully with the Denominators, thus, $2/20$, $3/20$, . . . of a pound.

The same we do also in the twelve parts of a shilling saying, 1 d. 2d. . . briefly with the Numerators onely, and not fully thus, $1/12$, $2/12$, . . ., of a shilling with the Denominator.

Later he sums up by saying (Ibid., p. 10.):

From this Abreviation of Fractions ariseth that late most useful invention of Decimall Arithmetique, where the Denominators are always 1 with Cyphers, 10, 100, 1000, 10000, &c. and therefore the Denominators are quite omitted (as alwayes certainly knowne being still more by one place than the figures of the Numerator) and the Numerators onely set down.

We need not stop here to give the various forms of the decimal separatrix, since that has been treated adequately above. Suffice it to say that the vertical stroke played

an important part as forerunner of a decimal separatrix in the writings of many prominent sixteenth-century authors already mentioned. Voigtel (No. 97, pp. 1-16.) who begins his Markscheide-Kunst with an explanation of 'so genannten Decimal-Rechnung', and Lucas (No. 58, pp. 224-231.) who devotes a chapter to 'Arithmetica Geometrica Decimalis', both still adhere to the sexagesimal exponents as their decimal notation, a slight deviation from the symbolism used by Stevin.

Like Stevin, Moore in his introduction to decimal arithmetic, writes a plea for the introduction of decimals into the systems of money, weights, and measures (No. 63, pp. 14-15.):

In practicall Arithmeticke, if the first institution had been in Decimalls, we had never been troubled with so many fractions; it were yet worthy the name of Reformation to cause the fractions of mony and weight to be altered: And as concerning the ease in measure, Surveyors and Land-meaters, who use the decimall chaine, and those who use a decimall foot, yard or scale can best certifie upon experience: If it had not been the Stationer's fault, I had long since published some Tables, after the most easie and methodicall way hitherto thought on, towards the ready obtaining the true place of any of the planets in this decimall way, which would have been a sufficient testimony of the ease thereof in Astronomicall calculations. Thus much for an Introduction into Decimall Arithmetick.

(F. Unger, Die Methodik der Praktischen Arithmetik, Leipzig, 1888, pp. 104-106, likewise quotes Christian Schessler, 1692, on the advantages of a decimal system.).

As regards the fundamental operations with decimals, we find that they differ little from our modern methods. Directions are usually given to proceed as with integers, and rules are set forth for determining the number of decimal parts in the result. Moore performs all the fundamental operations with decimals, and besides has a section on the reduction of fractions into decimals and vice versa, and gives reduction tables for money, weight, time, and measure. He applies decimals to proportions, compound interest, and logarithms. In fact, his entire work gives quite the modern treatment of decimals, 'for the great facility it brings with its practice in all parts of Arithmeticke, Astronomy, and the rest of the Mathematicks' (Ibid., p. 14.).

D. PROPORTIONS

As a general rule, proportions arise chiefly because of their use in the 'rule of three', so very popular in these early centuries. Some writers distinguish between a continued and discontinued proportion, as do Aurel (No. 3, f. 18, r.), Ghaligai (No. 38, f. 22, v.), Unicornio (No. 93, f. 156, v.), Alsted (No. 2, p. 53.), and Puig (No. 72, p. 31.). A continued proportion, a proportion whose two means are equal, usually termed a progression, does not concern us here so much as a discontinued or disjunct proportion. Very few authors treat the three kinds of proportion taught by Pythagoras, namely, arithmetic, geometric, and harmonic. The work of Bradwardin (No. 13.), however, is devoted chiefly to proportion in general.

Harmonic proportion, defined by $a:c::(a-b):(b-c)$, is rarely given attention in these arithmetic texts. Puig (No. 72, p. 33.) mentions it. Aurel (No. 2, ff. 15. v., 16, r.) gives the very interesting story of the invention of harmonic ratio by Pythagoras. Harmonic proportion as a combination of geometric and arithmetic proportions is illustrated by Aurel (No. 3, f. 18, r.) in the example 6.3.2., so that, according to the rule expressed above, $6:2::(6-3):(3-2)$. He is one of the few writers who treat the harmonic mean and give instructions how to find it. An explanation in ten lines of harmonic proportion may be found in Bassi (No. 6, p. 509), something given seldom (Noted also by Tropfke, v. III, 2nd. ed., p. 6.).

Although arithmetic proportion is mentioned by these same writers, and as well by Cardan (No. 16, Caput 45.), Alsted (No. 2, p. 53.), and Prestet (No. 70, p. 168.), yet little use is made of it. Of all these men, Prestet treats it at the greatest length. He speaks of it as 'l'egalité de deux différences', which we may express by the equation $a-b=c-d$. In the first theorem, he proves that in every arithmetic proportion the sum of the extremes is equal to the sum of the means, thus: $a+d=b+c$; and in the corollary (*Ibid.*, p. 169.), that in every continued arithmetic proportion the sum of the extremes is equal to the mean proportional taken twice.

Geometric proportion is an integral part of practically every early arithmetical text, and is given special treatment under the 'rule of three.' Micyllus (No. 60, p. 80.) refers to 'Regula Proportionum' and says it is called 'Regula De Tre' by the Italians. Others, like Suevus (No. 88,

p. 245.), Clavius (No. 25, p. 117.), Griminelli (No. 43, p. 131.), Ciacchi (No. 22, p. 121.), and Prestet (No. 70, p. 194.) also call it the 'golden rule', as the latter says, 'à cause du grand usage que l'on en fait.' Hodder (No. 51, p. 87.) remarks this:

As Gold transcends all other Metals, so doth this
Rule all other in Arithmetick.

The term 'golden rule' is also used by Recorde, Baker, Moore, and other early writers in English. Waser (No. 99, f. 48, v.) and Bohme (No. 7, p. 49.) add still another name, 'Regula Mercatorum', for the latter says:

Weil die Kauffleute von wegen ihres grossen Handels
dieselben nicht können ohne sein.

The great majority of the authors of those centuries speak of the 'rule of three' simple, and composée or composta. The latter is only an application of the 'rule of three' two or three times, and consists of five terms, of which the sixth may be found by dividing the product of the last three terms by that of the first two. The 'rule of three' may be direct and indirect (inverse or reciprocal), as illustrated by 2:3::4:6 and 2:6::4:3 respectively. In the former, the product of the means equals the product of the extremes according to Euclid, Prop. 19, Book 7, a fact enunciated again by Rudolff (No. 81, f. 37, v.), Clavius (No. 25, p. 119.), Alsted (No. 2, p. 58.), and Prestet (No. 70, p. 191.). In the latter, the first term is to the third as the fourth is to the second, so that in this case the product of the first two terms is equal to the product of the last two, according to Clavius (No. 25, p. 129.), Alsted (No. 2, p. 59.), Bassi (No. 6, p. 113.), and Prestet (No. 70, p. 193.).

The four common forms of writing a proportion, by inversion (conversa), alternation (permutata), composition (componendo, conjunta), and division (dividendo, disjunta), may be found complete in Cardan (No. 16, Caput 45.), Unicornio (No. 93, f. 156, v.), Puig (No. 72, p. 32.), and Prestet (No. 70, p. 192.). Clavius (No. 25, pp. 122, 123.) and Alsted (No. 2, p. 59.) mention mainly the first two, while Rudolff (No. 81, f. 37, v.), Regius (No. 76, f. 100, r.), and Vuelpius (No. 98, f. 43, v.) give only proportion by alternation. To the four forms mentioned above Unicornio

(No. 93, f. 157, r.) and Puig (No. 72, pp. 32, 33.) add proportion eversa expressed by $a:a - b::c:c - d$, proportion equal, $a:c::d:f$ obtained from $a:b::d:e$ and $b:c::e:f$. Puig (*Ibid.*, p. 33.) extends the list still further and illustrates as follows:

Ordinata, $b:a - b::d:c - d$, as 12.9.8.6,

Inordinata, $b:a - b::\frac{bc}{a-b}:c$, 12.9.24.6,

Extensa, $a:b::c:d::b:e::d:f$, 12.6.8.4.3.2,

Perturbata, $a:b::c:d::b:e::f:c$, 12.6.8.4.3.16.

Thirty-nine other properties of proportions follow in Puig (*Ibid.*, pp. 43-52.). Among these is included the Euclidean statement (*Ibid.*, p. 44.) that if $a:b::c:d::e:f$, then $(a + c + e):(b + d + f)::a:b$, also given by Chauvet (No. 21, f. 44, v.) and Prestet (No. 70, p. 188.).

Rudolff (No. 81, ff. 38, v., 39, r.) gives express directions for setting down the terms of a proportion and of finding any one missing term of a proportion. He follows this by giving rules for reduction of a proportion to simpler terms, which he accomplishes by dividing the first and the third, or the first and second terms by a common divisor, as illustrated below (*Ibid.*, f. 40, r.) in the proportion $32:36::18:x$:

32.	36.	18
8.	9.	18
4.	9.	9

This same procedure is used by Clavius, Alsted, Chauvet, and Monier.

The ordinary method of finding the mean proportional between two numbers is given by a large majority of authors, if mean proportion is treated at all. Puig (No. 72, p. 38.) finds the mean proportional between 24 and 6 in a second way, as illustrated below:

$$24 \div 6 = 4; \sqrt{4} = 2; \quad 24 \div 2 \text{ or } 6 \times 2 = 12.$$

E. POWERS

In the treatment of the last two sections we attempt to avoid infringing too much upon the domain of algebra. Operations with powers, outside of squaring and cubing, were not an integral part of the representative arithmetics of this period. Here we shall discuss powers only in so far as they were mentioned by the writers of arithmetic in connection with the extraction of roots. The formation of the names and symbols for exponents has been treated above.

Because it is a rare exception to find involution included among the fundamental operations, we quote its definition from Rahn (No. 73, p. 2.):

Involution is multiplying any quantity into its self, and that Product into the first quantity, and this last into the first, &c.

$$\begin{array}{r} 2 \quad . \quad 4 \quad . \quad 8 \quad \quad \&c. \\ \hline 2 \quad \quad 2 \quad \quad 2 \\ \hline 4 \quad \quad 8 \quad \quad 16 \end{array}$$

The significance of the term involution may be best gleaned from the following explanation on page 9:

Involution in Single Whole Quantities.

Its Sign is [⊙]

R. If a Quantity be involved or drawn into it self, or afterward into that Product, or again thirdly, into the last Product, &c., as manifold as is the Power, so great must the number be that is used to express it, being set after the Sign of Involution in the broader Column of the Margin.

.....

3 yz	abcd	1	
9 y ² zz	aabbccdd	2	1 ⊙ 2

.....

The Meaning is, let (3 yz) be a Line, this twice involved gives (9 yyzz) and is called a Quadratick Power or Square: Three times involved, it gives (27 yyyzzz) which is a Cubick Power, or Cube: Four times involved, it gives (81 y⁴z⁴) which is a Biquadratick, &c.

The ordinary definition of a numero quadrato, according to the definitions of the seventh and eighth books of Euclid, well given by Aurel (No. 3, f. 40.),

es un numero que procede de la multiplicaciõ de 2 numeros yguales en quâtidad y genero, como 3 y 3 multiplicados hazen 9, q̃ es numero quadrado: y el 3 es su rayz, o el 3 multiplicado en si (que es lo mesmo) y vernã 9, como cõsta por las definiciones del 8° del mesmo Euclides: diziẽdo: Numero quadrado es n° superficial de yguales lados. &c.

In a similar manner Prestet (No. 70, p. 64.) speaks of powers as 'Multiplications reïterées.'

The derivation of square numbers from odd numbers is found in Bassi (No. 6, p. 526.):

Li numeri quadrati hanno l'origine dalli numeri dispari, che derivano da un 'unità, sequitando in continua Proportione, che è la seconda spetie della Progressione Arimmetica, come saria 1.3.5.7.9.11.13.15, & altri simili, onde se l'I si congiungera con il 3 sequente, componerassi 4 secondo numero quadrato, e così congiunti insieme li tre numeri dispari 1.3.5, seranno 9, che è il terzo numero quadrato, similmente congiungendo insieme 1.3.5.7, daranno 16, per lo quarto numero quadrato, e seguirassi con tal modo ne gli altri.

He follows this later with the example:

. . . la radice di 36, sarà 6, perciò il numero quadrato 36 è composto di sei numeri dispari, cioè 1.3.5.7.9.11, li quali raccolti insieme, fanno 36.

Aurel (No. 3, f. 40.) and Puig (No. 72, pp. 3, 16, 321.) likewise give the composition or generation of square numbers. The latter calls attention to the fact that a square number does not end in 2, 3, 7, or 8.

We may mention a few of the more important writers who give tables of the squares and cubes of the first 9 digits: Aurel (No. 3, f. 41.), Clavius (No. 25, pp. 513, 517.), Alsted (No. 2, p. 74.), and Blondel (No. 9, p. 116.). Lucas (No. 58, pp. 205-214.) extends the table so as to include numbers up to 500. Similar tables of higher powers, as a convenience for extracting higher roots, are given by

Moore (1660), as noted previously, and also by Bassi (No. 6, pp. 522, 524.), Puig (No. 72, pp. 327, 330.), and Prestet (No. 70, fold. table opposite p. 100.).

Whereas Tropicke (v. II, 3d. ed., p. 158.) has cited Girard as affording a single instance of the use of fractional exponents, he has neglected to include him as employing zero exponents. Girard (No. 39, f. 9, r.) writes:

. . . toutefois le (0) est de signification finie,
suivant un nombre, comme 18 (0) est 18 precisement,
car c'est 18 nullement indefiny, quant à (1) 18, c'est
le mesme que 18 (0) car ils s'accordent là.

According to his notation, 18 (0) stands for $18x^0$, and (1) 18 for the first root of 18, both of which equal 18.

F. ROOTS

Although the extraction of roots is not always considered a fundamental operation in the early arithmetics, it is generally given special treatment. Of the many texts examined, we find among the works examined, its discussion entirely missing only in the works of Rudolff (No. 81.) (1532), Felner (No. 30.) (1525), Cantone (No. 15.) (1606), and Behme (No. 7.) (1695). It is noteworthy here, too, that Vuelpius (No. 98.) (1544) does not attempt to take the square root of a numerus non quadratus.

Rahn's definition of the process of root extraction (No. 73, p. 2.) is worth quoting:

Evolution is the discovery of the Ingredients of the involved quantity.

$$\begin{array}{r} 8 \quad 6 \quad 4 \quad 2 \\ \underline{2} \quad \underline{2} \quad \underline{2} \quad \underline{2} \\ 6 \quad 4 \quad 2 \quad 0. \end{array}$$

Aurel (No. 3, f. 40, v.) refers to the sixth book of Euclid when he writes that square root is to find the mean proportional between unity and the proposed number.

Tropicke (v. II, 3d. ed., p. 175.) mentions the method of approximation in the extraction of roots as explained by Clavius (1583). We find that Clavius (No. 25, pp. 274-278.), in the later editions examined, explains three old methods of approximation. Two of these methods he obtains from

Ptolemy's *Almagest*, finding one value less and one greater than the true root by the further development of the formula

$$\text{of Archimedes, } a \pm \frac{b}{2a} > \sqrt{a^2 \pm b} > a \pm \frac{b}{2a \pm 1}.$$

A part of the above formula for approximating the root of a number which is an imperfect power, namely,

$$\sqrt{a^2 + b} = a + \frac{b}{2a + 1}, \text{ was used by Aurel (No. 3, f. 42, r.),}$$

Micyllus (No. 60, p. 61.), Steinmetz (No. 86, f. 58.), Bonvicino (No. 12, p. 31.), Puig (No. 72, p. 321.), and Blondel (No. 9, p. 145.). Puig explicitly gives the reason for the form of the denominator of the remainder. In a similar way Monier (No. 62, pp. 426, 463.) approximates the cube root of a number by the formula,

$$\sqrt[3]{a^3 + b} = a + \frac{b}{3a + 3a^2}. \text{ (Günther, } \underline{\textit{Geschichte des Mathemat-}}$$

ik, Leipzig, 1908, v. I, p. 343, refers to the use of these formulas by Ortega in 1512.).

Both Clavius (No. 25, pp. 274-278) and Prestet (No. 70, pp. 102, 103.) refer to an old method of approximation which corresponds to our modern method of using decimal fractions. By annexing three pairs of zeros to 20, its square root is

given by the former as $4 \frac{472}{1000}$, whereas Prestet would give the result as $\frac{4472}{1000}$. It is worth adding that Scheubel

(No. 83, pp. 64, 65.) devotes considerable space to this decimal method of approximation. Voigtel (No. 97, pp. 5-12) finds the square roots of decimal numbers themselves.

Our modern method of obtaining the square root by use of the principle $(a + b)^2$ is well illustrated in Rahn (No. 73, p. 15.). Authors like Micyllus (No. 60, p. 60.), Chauvet (No. 21, f. 193, r.), and Blondel (No. 9, pp. 119-121.) take pains to illustrate this principle geometrically according to the second book of Euclid. (L. L. Jackson, *Sixteenth Century Arithmetic*, pp. 122, 123, quotes in detail the explanation of this method given by Trenchant in his *L'Arithmetique*, 1578.).

As usual we find an extended treatment of higher roots in Puig (No. 72, pp. 322-332.) and Prestet (No. 70, pp. 83-90.). The former takes roots as high as the fifth by use of the

binomial coefficients; Prestet, as also Graffenried (No. 42.), up to the ninth; but they suggest the shorter method for the fourth, sixth, and ninth roots by use of the law $(a^{\frac{1}{m}})^{\frac{1}{n}} = a^{\frac{1}{mn}}$.

So far as the treatment of operations with radicals is concerned, this properly belongs in the realm of algebra.

* * * * *

BIBLIOGRAPHY

1. Author: Joannes Alexander, of Bern.
 Title: Synopsis algebraica, opus posthumum Johannis Alexandri, Bernatis-Helvetii. In usum scholae mathematicae apud Hospitum-Christi Londinense, à Carolo II fundatae, editum.
 Description: t.-p., (2), 204 pp., (2), diagrs., 17½ cm., London, 1693, (U.M.).
 Remarks: Another edition: Ed. 2, emendatior. Cui adjungitur appendix per Humfredum Ditton. London, 1709, (U.M.). Cajori speaks of an English translation by Sam.Cobb at London in 1709.

2. Author: Johann Heinrich Alsted.
 Title: Methodus admirandorum mathematicorum Novem libris exhibens universam Mathesin.
 Description: t.-p., 456 pp., diagrs., 12 cm., no app., Herborn, 1641 (3d. ed.), (U.M.).
 Remarks: Other editions: 1st. ed., Herborn, 1613, (D.E.S.); 2nd. ed., ib., 1623; 4th. ed., ib., 1657. Wrote also Elementa mathematica, Frankfurt, 1611.

3. Author: Marco Aurel, el Aleman, 16th. cent.
 Title: Libro primero, de arithmetica, Algebratica, enel qual se contiene el arte Mercantiuol, con otras muchas Reglas del arte menor, y la regla del Algebra, vulgarmente llamada Arte mayor, o Regla de la cosa: sin la qual no se podra entender el decimo de Euclides, ni otros muchos primores, assi en Arithmetica como en Geometria:...
 Description: t.-p., 4 ff. unnumb. + 140 numb. = 144 ff., 19.3 cm., Valencia, 1552, (U.M.).
 Remarks: There was no other edition.

4. Author: Ventura de Avila.
 Title: Elementos de algebra, o sea reglas generales para encontrar lo que vale la incognita en las equaciones de el primero, y segundo grado, en quienes no haya termino irracional, y resolucion de setenta y quatro problemas, distribuido todo en veinte y tres Dialogos de suerte, que en otros tantos dias los puede comprehender el que se halle impuesto en los setenta y siete Dialogos dados à luz por el Autor de los presentes.....

- Description: t.-p., (4), 107 pp., 3 fold. plates,
14.9 cm., Barcelona, n. d., (U.M.).
Remarks: The work in hand bears no date; probably of
the late seventeenth century.
5. Author: Giulio Bassi, 17th cent.
Title: Dell' Arimmetica pratica di Giulio Bassi.....
Libri VII.
Description: 1 p. l., t.-p., (12), 492 pp., 31 cm.,
Piacenza, 1645, (U.M.).
Remarks: Another work, Abbaco, appeared in 1791.
6. Author: Same as above.
Title: Arimmetica, e geometria practica del dottor
Giulio Bassi Piacentino.....Libri Otto.
Description: 1 p. l., t.-p., (12), 616 pp., 31 cm.,
Piacenza, 1666, (U.M.).
Remarks: A rev. ed. appeared ib., 1765, (U.M.).
7. Author: Caspar Behme (?).
Title: Arithmetica oder Rechen-kunst.
Description: t.-p., (4), 66, sub. t.-p., 38 pp., 15½ cm.,
Dantzig, 1695, (U.M.).
8. Author: Joannes Martinus Blasius.
Title: Liber arithmetice practice astrologis Phisicis
et calculatoribus admodum utilis.
Description: 26 ff. unnumb., 64-66 ll., printed in
double columns, 28 cm., Paris, 1513, (G.A.P.).
Remarks: Other editions: Paris, 1514; ib., 1519,
(G.A.P.); ib., 1526.
9. Author: Francois Blondel.
Title: Cours de Mathematique, qui contient la mathema-
tique en general, la geometrie speculative, et la
geometrie pratique..... Seconde Edition, augmentée
& corrigée.
Description: t.-p., (22), 198, t.-p., (5), 236 pp.,
27 cm., Amsterdam, 1699 (2nd. ed.), (U.M.).
Remarks: 1st. ed., Paris, 1683 (2 vol.). Blondel also
wrote other works on architecture, fortification, and
the Roman calendar.

10. Author: Johann Böschensteyn.
Title: Ein New geordnet Rechenbiechlin....
Description: 24 ff. unnumb., 30 ll., 19.2 cm.,
Augsburg, 1514, (G.A.P.).
Remarks: Other editions: Augsburg, 1516; ib., 1518.
His son, Abraham, wrote Ein nützlich Rechenbuchlin,
s. l., 1536, (G.A.P.). This work is very rare.
11. Author: Rafael Bombelli.
Title: L'algebra opera Di Rafael Bombelli.....
Description: 1 p. l., t.-p., (56), 650 pp., (4),
20 $\frac{1}{2}$ cm., Bologna, 1579, (U.M.).
Remarks: 1st. ed., Venice, 1572, (D.E.S.).
12. Author: Valeriano Bonvicino.
Title: Matematiche discipline.....aritmetica.....
Description: t.-p., (12), 201 pp., (2), illus.,
22 cm., Padua, 1665, (U.M.).
Remarks: Another ed., Padua, 1666. Riccardi gives
this edition only, and says that the Bib. militaire
(p. 87) registers an ed. of 1665. We have the 1665
ed. under direct examination.
13. Author: Thomas Bradwardin.
Title: Arithmetica thome bravardini.....
Description: 6 ff. unnumb., 61 ll., printed in double
columns, 27.8 cm., Paris, c. 1510, (G.A.P.), a
unique copy.
Remarks: Editions: Paris, 1495; ib., 1496; ib., 1498;
ib., c. 1550; ib., 1502; Valencia, 1503; Paris,
1504; ib., 1505, (C.U.); ib., 1512; ib., 1530;
Wittenberg, 1534; ib., 1536. Probably one of the
oldest arithmetics written by an English mathemati-
cian after Baeda and Alcuin.
14. Author: Philippi Calandri.
Title: Pictoragas arithmetice introductor.
Description: 104 ff. unnumb., 9-26 ll., printed in
double columns, 10.5 x 14 cm., Florence, 1491, (U.M.).
Remarks: Another ed., Florence, 1518, (U.M.). It is
the first arithmetic to give division a danda.

15. Author: Oberto Cantone.
 Title: L'uso pratico dell' aritmetica e geometria
 d'Oberto Cantone... seconda impressione....
 Description: 1 p. l., t.-p., (8), 292 pp., 19 $\frac{1}{2}$ cm.,
 Naples, 1606, (U.M.).
 Remarks: 1st. ed. Naples, 1599, (U.M.). There are
 also the 1609 and 1612 editions.

16. Author: Hieronimo Cardan.
 Title: Hieronimi C. Cardani medici Mediolanensis,
 practica arithmetice, & Mensurandi singularis....
 Description: 1 p. l., t.-p., unpagged, illus., 16 $\frac{1}{2}$ cm.,
 Milan, 1539, (U.M.).
 Remarks: Besides other works, Cardan published the
 famous Ars Magna in 1545, which was reprinted in
 Cardan's Opera, 1663, (U.M.).

17. Author: Pietro Antonio Cataldi.
 Title: Nuova algebra proportionale.....
 Description: 1 p. l., t.-p., (4), 51 pp., diags. on
 2 pl., 29 $\frac{1}{2}$ cm., Bologna, 1619, (U.M.).
 Remarks: Earlier edition, 1610. Cataldi also wrote
Due Lettioni, 1577, (U.M.), and 1613, (U.M.), be-
 sides several other works on arithmetic, algebra,
 and geometry. At Bologna were also published Prima
Parte della Pratica Arithmetica, 1602, (G.A.P.), and
Seconda Parte....., 1606, (G.A.P.).

18. Author: Same as above.
 Title: Algebra applicata.....di Pietro Antonio Cataldi

 Description: t.-p., (4), 88 pp., diags., 31 cm.,
 Bologna, 1622, (U.M.).

19. Author: Pietro Cataneo, 16th cent.
 Title: Le pratiche delle due prime Mathematiche di
 Pietro de Catani da Siena.
 Description: 64 l. incl. diags., 20 $\frac{1}{2}$ cm., Venice,
 1546, (U.M.).
 Remarks: 2nd. ed., 1559, (U.M.). There also are works
 on architecture, 1554, (U.M.), and 1567, (U.M.).

20. Author: Joannis Ceva.
 Title: Opuscula Mathematica.
 Description: t.-p., 2 p. l., 53 pp., (2), 8 fold.
 tables, 26 cm., Milan, 1682, (D.E.S.).
 Remarks: Ceva also wrote geometrical works.

21. Author: Jacques Chauvet.

Title: Methodiques Institutions de la vraye et parfaite arithmetique de Jacques Chauvet, diuisee en six parties. Reveue, corrige'e, et amplifie'e d'Exemples Geometriques,.....

Description: 2 p. l., t.-p., (8), 255 numbered l. (i.e., 510 p.), (10), 17.2 cm., Rouen, 1627, (U.M.).

Remarks: This work was a revision, correction, and amplification by P. Taillefer of Chauvet Champenois' Les Institutions of 1578, (G.A.P.). D. E. Smith's Rara Arithmetica says there was no other edition, and does not refer to this revised work. A still later ed. appeared at Rouen, 1648, (U.M.). Chauvet also wrote several works on geometry.

22. Author: Giuseppe Ciacchi.

Title: Regole generali d'abbaco.....

Description: 1 p. l., t.-p., (16), 398 pp., (1), illus., fold. tab., diags., 16 cm., Florence, 1675, (U.M.).

Remarks: Practical geometry is treated on the last 50 pages.

23. Author: Pedro Ciruelo.

Title: Tractatus arithmethice practice.....

Description: 27 pp., tables, 1 diag., 20 cm., Paris, 1505, (U.M.).

Remarks: Other Paris editions: 1495, 1509, 1513.

Ciruelo also wrote Cursus quattuor mathematicarum artiū liberaliū, which had nine editions. He also edited Bradwardin's arithmetic, 1495, and 1502.

24. Author: Christopher Clavius.

Title: Christophori Clavii Bambergensis e Societate Jesu Epitome arithmeticae Practicae.....

Description: 1 p. l., t.-p., 323 pp., (13), 16½ cm., Rome, 1585, (U.M.).

Remarks: 1st. ed., ib., 1583; other Latin editions: 1584, (G.A.P.), 1592.

25. Author: Same as above.

Title: Arithmetica prattica composta dal molto Rever. Padre Christoforo Clauio Bambergense della Compagnia di Jesu. Et tradotta da Latino in Italiano dal Signor Lorenzo Castellano.....

Description: 1 p. l., t.-p., 286 pp., (16), 16.3 cm., Rome, 1586, (U.M.).

- Remarks: Other Italian translations: ib., 1602, (G.A.P.); ib., 1618, (U.M.); ib., 1626, (D.E.S.); Venice, 1656, (D.E.S.); Venice, 1686, (U.M.). In 1611 appeared Opera omnia; in 1612, his collected works in 5 volumes at Basel entitled Opera mathematica, (U.M.). Clavius also wrote works on algebra, geometry, the calendar, and the astrolabe.
26. Author: Johan Coutereels.
 Title: Arithmetica.
 Description: t.-p., 1 p. l., 528 pp., diags., 16 cm., Middleburg, 1658, (U.M.).
 Remarks: In the preface Coutereels tells of previous editions: 1st. ed., 1599, 1606, 1610, and 1620, (G.A.P.). Also 't konstigh Cyffer-Boek, Utrecht, 1690, and Het Konstich Cyffer-Boek, Middleburg, 1626, (G.A.P.).
27. Author: Sebastian Curtius.
 Title: Arithmetica Practica, das ist: Ein Newes Wolgegründetes Rechenbuch.
 Description: t.-p., unpag., 20 cm., Leipzig, 1619, (D.E.S.).
 Remarks: At the end he speaks of examples taken from his 1600 Schuelrechenbuchleins, and later from the 1610 ed.
28. Author: Rene Descartes.
 Title: Geometria, à Renato Des Cartes, anno 1637 Gallicè edita;.....Operâ atque studio Francisci à Schooten,.....
 Description: 1 p. l., t.-p., (12), 336 pp., (2), 22½ cm., Lugduni Batavorum, 1649, (U.M.).
 Remarks: His 1st. ed. was in 1637. Another Latin ed., 1659-1661, 1683, (U.M.); French ed., 1886, (U.M.); English ed., 1925, (U.M.).
29. Author: Francesco Feliciano.
 Title: Libro di Arithmetica e Geometria speculatiua e praticale:..... Nouamente stampato.
 Description: t.-p., 159 pp., diags., 22 cm., Venice, 1550, (U.M.).
 Remarks: Substantially the same as the 1st. ed., Venice, 1526, (G.A.P.). The work is a revision of the author's Libro de abaco, which was first pub. in

Venice, 1517. Other editions: ib., 1527, 1536, (G.A.P.), 1545, (G.A.P.), 1550, (G.A.P.), 1560, (G.A.P.), 1560, 1561, ib., 1563, (U.M.), Verona, 1563, (G.A.P.), Venice, 1570, (G.A.P.), Verona, 1602, and other seventeenth-cent. editions, Scala grimaldelli, Padua, 1629, (G.A.P.), (D.E.S.), 1669, (U.M.), and 1692, (U.M.).

30. Author: Nicolaus Felner von Winnsheim.
 Title: Ein New Behenndes und ganz gründtlichs Rechen-
 bucklin auff der Linien und Federn.
 Description: t.-p., unpag., 14½ cm., Erfurt, 1535,
 (U.M.).
 Remarks: The only copy known. Not mentioned by Cajori,
 Tropfke, Smith, DeMorgan, and Jackson.
31. Author: Conrad Feme.
 Title: Eyn gut new rechenbuchleyn.
 Description: Sm. 8, 11 ll., diagrs., Erfurt, 1523,
 (G.A.P.).
 Remarks: Of the utmost rarity.
32. Author: Giuseppe Figatelli.
 Title: Trattado Aritmetico di Giuseppe Maria Figatelli
 In questa seconda impressione aggiuntoui
 l'algebra.
 Description: 6 p. 1., 251, 8, (6) pp., diagrs., 20½
 cm., Venice, 1678, (U.M.).
 Remarks: 1st. ed., 1664. Later editions: 1680, 1686,
 1692, (U.M.), 1699, (U.M.), a 6th impression, s.a.,
 1737, (U.M.), 1738, 1778, 1791, 1797. Riccardi does
 not mention the above 2nd. impression.
33. Author: Oronce Finé.
 Title: Orontii Finaei delphinatis..... De arithmetica
 practica libri quatuor:.....
 Description: (8) pp., 72 numb. 1. (i.e., 144 p.)
 incl. tables, 22 cm., Paris, 1555, (U.M.).
 Remarks: 1st. work, Orontii Finei Delphinatis, Proto-
mathesis, Paris, 1532, (U.M.). A Latin ed. is de-
 scribed above. Other Latin editions: Paris, 1535,
 (N.Y.P.), 1542, (G.A.P.), 1544, (G.A.P.), 1554 Finé
 also wrote geometrical works.

34. Author: Same as on preceding page.
 Title: Opere di Orontio Fineo del Delfinato: diuise in cinque parti; arimetica,....tr. da Cosimo Bartoli.....
 Description: 8 p. l., 81, 126, 88, 18 numb. l., 2 l. illus., diagrs., 20½ cm., Venice, 1587, (U.M.).
 Remarks: Another Italian translation, Venice, 1670, (U.M.).
35. Author: Lorenzo Forestani.
 Title: Pratica d'Arithmetica, e Geometria.....
 Description: t.-p., 10 p. l., 344 ff., (1), illus., diagrs., 20 cm., Venice, 1603, (U.M.).
 Remarks: An ed. at Siena, 1682: Di nuouo Ristampa, e con somma diligenza Ricorretta, (U.M.).
36. Author: Pierre Gautruche.
 Title: P. Petri Galtruchii.....Mathematicae totius,...
 Description: t.-p., (6), 304 pp., (4), 15 cm., Cambridge, 1668, (U.M.).
 Remarks: Gautruche also wrote Philosophiae ac mathematicae totius institutio....., Vienna, 1661, (U.M.), and Cadomi, 1665, (U.M.), besides works on metaphysics and poetical history.
37. Author: Gemma Reinerus, Frisius.
 Title: Arithmeticae practicae Methodus facilis per Gemmam Frisium, Medicum ac Mathematicum, iam recens ab ipso autore emendata & multis in locis insigniter aucta....
 Description: 96 numb. l. incl. tables, diagrs., 17 cm., Paris, 1549, (U.M.). With this are bound Köbel, Jacob. Astrolabii declaratio. 1550; Cosmographiae introductio. 1550.
 Remarks: 1st. ed., Antwerp, 1540, (N.Y.P.). The editions Witebergae, 1550, (U.M.), and Paris, 1550 (U.M.) are substantially the same. Smith's Rara cites 59 or more editions in the 16th cent. besides numerous editions after 1600. We also have these editions: Venice: 1567 (U.M.), and 1576, (U.M.).
38. Author: Francesco Ghaligai.
 Title: Practica d'arithmetica di Francesco Ghaligai Fiorentino. Nuouamente Riuista, & con somma Diligenza Ristampata.

Description: t.-p., (2), 114 numb. l., 21 cm.,
Florence, 1552, (U.M.).

Remarks: The 1st. ed. was issued in 1521 under the
title Summa de arithmetica. We also have a 1548
ed., (U.M.).

39. Author: Albert Girard.

Title: Invention nouvelle en l'Algebre. Reimpression
par Dr. D. Bierens de Haan.

Description: 4 p. l., t.-p., 64 unnumb. pp., diagrs.,
21½ cm., Leyden, 1884, (U.M.).

Remarks: This is a reprint of the original ed.,
Amsterdam, 1629. Girard also revised a work of
Samuel Marolois in 1628, (U.M.), and wrote on the
mathematical works of Simon Stevin, 1634.

40. Author: Henricus Glareanus.

Title: De Vl. Arithmeticae practicae speciebus.

Description: 2 ff. unnumb. + 21 numb., 29 ll., 17 cm.,
Paris, 1543, (G.A.P.).

Remarks: Other editions: Freiburg, 1543; Cracow,
1549; Freiburg, 1550, (G.A.P.); Paris, 1551,
(G.A.P.); Freiburg, 1555; ib., 1558; Paris, 1558.
Glareanus published other works besides this.

41. Author: Girjka Gorla Z Gorlssteyna.

Title: Wyliffthenow Starém Héfté Praž kém v Girijska
Czerneho.

Description: 9 ff. unnumb. + 89 numb., 21 ll., 15.9
cm., Czerny, 1577, (G.A.P.).

Remarks: A very rare arithmetic in the Polish language.

42. Author: Johan Rudolff von Graffenried.

Title: Arithmeticae Logistica Popularis Libri IIII.

Description: t.-p., 28 p. l., 704 pp., (15), 20 cm.,
Bern, 1618, (D.E.S.).

Remarks: This volume is from the library of M. Chasles.

43. Author: Domenico Griminelli.

Title: Novissima prattica d'arithmetica mercantile
Composta dal R. Don Domenico Griminelli....

Description: t.-p., 4 p. l., 420 pp., (4), 16½ cm.,
Rome, 1656, (U.M.).

Remarks: We also have a 2nd. ed., 1670 (U.M.), and
Riccardi mentions a probable 3d. ed.

44. Author: Cardinaels Sybrand Hansz.
Title: *Arithmetica, ofte Reecken-Konst.*
Description: t.-p., unpagcd, 15½ cm., Amsterdam, dates on t.-p. of its 4 books, 1665, 1658, 1661, 1660, respectively, (G.A.P.).
45. Author: Thomas Harriot.
Title: *Artis analyticae praxis*,.....
Description: 1 p. l., t.-p., (4), 180 pp., 1 l., 34½ cm., London, 1631, (U.M.).
Remarks: Harriot also wrote historical works.
46. Author: Andreas Helmreich.
Title: *Rechenbuch*....
Description: 111 ff. unnumb. + 1 blank, 27-33 ll., 19.3 cm., Eisleben, 1561, (G.A.P.).
Remarks: Republished at Leipzig in 1588, (G.A.P.), 1595, and possibly in 1596. Also wrote Ein new Visier büchlein, 1557, (U.M.).
47. Author: Johann Hemeling.
Title: *Arithmetica Historia oder Historische Rechenkunst*....
Description: 8 p. l., 148 (i.e., 198) pp., 15½ cm., Hanover, 1667, (U.M.).
Remarks: Pp. 192-198 erroneously numbered 142-148.
48. Author: Same as above.
Title: *Johannis Hemelingii Selbstlehrende Rechneschul oder Selbstlehrendes Rechenbuch*....
Description: 13 p. l., t.-p., 658 + pp., 17½ cm., 20 blank leaves interspersed, Hanover? 1663? (U.M.).
Remarks: There is also a revised edition, Johann Hemelings Kleines rechenbuch.... by E. A. Talgener, Hanover, 1817, (U.M.).
49. Author: Pierre Herigone.
Title: *Cursus mathematicus*,.....
Description: 1 p. l., t.-p., (80), 983 pp., (21), diagrs., 18 cm., Paris, 1634, v. 1, (U.M.).
Remarks: 2nd. ed., 1644, (U.M.).
50. Author: James Hodder.
Title: *Hodder's Decimal Arithmetick*.
Description: t.-p., 5 p. l., 157 pp., (3), 15 cm., London, 1671, (G.A.P.).

51. Same as on preceding page.

Title: Hodder's Arithmetick..... Eighteenth Edition...
...revised, augmented and.....amended.....by Henry
Mose.

Description: Photograph, t.-p., 5 p. l., 216 pp.,
14.5 cm., London, 1693, (N.Y.P.).

Remarks: This is the first arithmetic separate in
America. There is also a 1719 ed. (N.Y.P.). On the
inside of cover is written: "Not mentioned by
DeMorgan; he only says the 1st. was in 1661, the 3d.
in 1664, and the 9th. in 1672." There is a 15th.
ed., 1685, (U.M.), and a 22nd. ed., 1702, (U.M.).

52. Author: Johannes Husswirt.

Title: Enchiridion novus Algorismi.....

Description: 2 p. l., t.-p., 39 unnumb. pp., blank ll.
bound at end of book, 21½ cm., Cologne, 1501, (U.M.).

Remarks: Other editions: 1503, 1504, 1507, 1554; and a
French ed., (Chasles). Later published with histor-
ical notes by Professor Wildermuth, at Tübingen,
1865, (D.E.S.).

53. Author: Sieur C. Irson.

Title: Arithmetique pratique et raisonnée.

Description: t.-p., 9 p. l., 557 pp., (2), 25 cm.,
Paris, 1692, (N.Y.P.).

54. Author: Simon Jacob.

Title: Ein New und Wolgegründt Rechenbuch auff den
Linien und Ziffern samt der Welschen Practic....zum
zweytenmal getruckt.

Description: 11 p. l., t.-p., 349 pp., (1), diags.,
20½ x 17 cm., Frankfurt, 1612, (U.M.). Preface dat-
ed 1552; colophon, 1600.

Remarks: 1st. ed., 1560; other editions: 1565,
(G.A.P.), 1569, 1600, (G.A.P.), and in the 17. cent.
Jacob also wrote Rechenbüchlein auf den Linien und
mit Ziffern, Frankfurt, 1557, 1574, 1589, 1590, 1599,
(C. U.). The two works may be the same.

55. Author: Heino Lambeck.

Title: Heinonis Lambecii vermehrtes und verbessertes
Rechen-buch.'....Neue Auflage. Vermehrt mit der So-
lution der algebraischen Beschluss-Aufgabe des
Johann Hinrich Wohlgemuths.

Description: t.-p., 324 pp., 17½ cm., Stade, n. d., (U.M.).

Remarks: No date on t.-p. According to Poggendorf, he died in 1661. Murhard gives the date 1661 for this work. The present writer, in her research of this book, came across a division problem on page 44 of which the quotient gave 1661 as the year in which Lambeck's Rechen-buch was printed.

56. Author: Giorgio Lapazzaia.

Title: Libro d'arithmetica e geometria dell' Abbate Giorgio Lapazzia.

Description: 4 p. l., t.-p., 295 (2) pp., 20 cm., Naples, 1601, (U.M.).

Remarks: Lat. ed., 1566; Ital. editions; 1566. (G.A.P.), 1569, (G.A.P.), (D.E.S.), 1575, (G.A.P.), 1590, (G.A.P.), besides several editions after 1600. Riccardi records an ed. in 1784.

57. Author: Francois Legendre.

Title: L'arithmetique en sa perfection.....neuvieme edition.

Description: t.-p., 4 p. l., 384 pp., 25 cm., Paris, 1690, (U.M.).

Remarks: We also have the 11th.ed., 1700, (U.M.).

58. Author: Lucas à S. Edmundo.

Title: Arithmeticus practicus,.....

Description: t.-p., (16), 346 pp., (2), 14½ cm., Tyrnaviae, 1697, (U.M.).

59. Author: Faentino, Francesco Manelli.

Title: Opera d'arithmetica e geometria....

Description: t.-p., 3 p. l., 176 pp., port., diagrs., 23 cm., Bologna, 1659, (U.M.).

Remarks: This work is bound with Trigonometria by Cavalieri, 1643, and is followed by a work on the compass by Galilei, 1640.

60. Author: Jacobus Micyllus.

Title: Arithmeticae logisticae libri duo,.....

Description: t.-p., 8 p. l., 294 pp., (9) pp. incl. tables, diagrs., 15½ cm., Basel, 1555, (U.M.).

61. Author: Johannes Monhemius.
Title: *Methodus arithmetices computatoriae*.....
Description: 22 ff. unnumb., 25-27 ll., 15cm.,
Düsseldorf, 1561, (G.A.P.).
Remarks: Almost unknown to arithmeticians.
62. Author: Jean Monier de Clairecombe.
Title: *Nouvelle pratique d'arithmetique*.....
Description: t.-p., 8 p. l., 466 pp., (16), 13½ cm.,
Amsterdam, 1693, (U.M.).
63. Author: Jonas Moore.
Title: *Moor's Arithmetick*. In two books. The first
treating of the Vulgar Arithmetick.... The Second
of Arithmetick in Species or Algebra.... To which
are added two Treatises:.....
Description: 7 p. l., (455) p., front. (port.), diags.
on 7 pl., (partly fold.) tables, 17 cm., various pag-
ing; the 2nd. book and the 2 treatises each have a
special t.-p., London, 1660, (U.M.).
Remarks: 1st. work, *Moor's Arithmetick*, 1650; 2nd. ed.,
Moor's Arithmetick in two books, 1660, (U.M.); 3rd.
ed., *Moor's Arithmetick in four books*, 1688,
(G.A.P.). Moore also wrote *Mathematical Compendium*,
(U.M.), London, 1674, 1681, (D.E.S.), and *A New Sys-
teme of the Mathematicks*, London, 1681, in 2 volumes,
(U.M.).
64. Author: John Napier.
Title: *Raddologia Ouero arimmetica virgolare*....Traddo-
tore dalla Latina nella Toscana lingua.....
Description: t.-p., 1 p. l., (14), 269 pp., (1) p.
incl. illus., 7 pl. (6 fold.) tables, diags.,
16 cm., Verona, 1623, (U.M.).
Remarks: 1st. ed., 1617, (D.E.S.), (N.Y.P.); other edi-
tions; 1626, 1628, (N.Y.P.). Napier also wrote *De-
scriptio*, 1618, and *Constructio*, 1619, 1620. A ms.
of his on algebra was printed in 1839.
65. Author: Joannes Noviomagus, pseud., otherwise, Jan van
Bronkhorst.
Title: *De numeris libri II*.....
Description: 2 v. in l., illus., 15½ cm., Cologne,
1544, (U.M.).
Remarks: Other editions: 1539, (G.A.P.), (C.U.), 1551.
We also have this same work 'esposti e illustrati

dal prof. G. Frizzo,, Verona,Drucker,
1901, (U.M.).

66. Author: Juan de Ortega.

Title: Suma de Arithmetica: Geometria Practica utilis-
sima.....

Description: 2 ff. unnumb. + 114 numb., 32-38 ll.,
30.2 cm., Rome, 1515 (G.A.P.), (D.E.S.).

Remarks: 1st. ed. Barcelona, 1512; other editions:
Lyons, 1512; ib., 1515; Messina, 1522; 1534; Seville,
1536; ib., 1537; Paris, 1540 (?); Seville, 1542,
(G.A.P.); ib., 1552 (G.A.P.), (D.E.S.): s. l. (?),
1552; Granada, 1563.

67. Author: Georg Peurbach.

Title: Elementa Arithmetices.....

Description: Special t.-p. (32nd. leaf)., 16 cm.,
Venice, 1539, (U.M.).

Remarks: 1st. ed., 1492; other editions: 1500, 1503,
1507, 1510, 1511, 1511 (earlier), 1511, 1512, 1513,
1515, (G.A.P.), 1520, 1534, 1536, 1538, 1539, (U.M.),
1544. The above work is contained in the 2nd. half
of Elementa Geometriae by Johann Vogelin. Peurbach
wrote many works on geometry and astronomy.

68. Author: Giovanni Francesco Peverone.

Title: Due brevi e facili trattati il primo d'Arith-
metica: l'altro di Geometria:.....

Description: t.-p., 132 pp., (2), illus. (incl.
diags.), 23½ cm., Lyons, 1558, (U.M.).

Remarks: There is also a 1581 ed. published at Lyons,
(G.A.P.).

69. Author: Monte Regal Piedmontois.

Title: Invention nouvelle.....

Description: Printed in double columns, 27-28 ll.,
9.8 cm., Lyons, 1585, (G.A.P.).

Remarks: This book is unknown to most bibliographers.

70. Author: Jean Prestet.

Title: Elemens des mathematiques, ou, Principes
generaux de toutes les sciences,.....

Description: t.-p., 8 p. l., 418 pp., 2 fold. tables,
25½ x 19½ cm., Paris, 1675, (U.M.).

Remarks: There is a later work of 1689 in 2 volumes.

71. Author: Prosdocimo de Beldamandi.
Title: Algorismus de Integris.
Description: 44 ff. unnumb., 30 ll., 14.5 cm., Venice, 1540, (G.A.P.).
Remarks: 1st. ed., Padua, 1483, (G.A.P.). From a note in the Venice ed. of 1540 it appears that it was written in 1410.
72. Author: Andres Puig.
Title: Arithmetica especulativa, y practica, y arte de algebra.....
Description: t.-p., 8 p. l., 576 pp., (6), 1 l., 19 cm., Barcelona, 1672, (U.M.).
73. Author: Johann Heinrich Rahn.
Title: An Introduction to Algebra, Translated out of the High-Dutch into English by Thomas Brancker.
Description: t.-p., 1 p. l., (5), 198 pp., 50 pp., 20 cm., London, 1668, (U.M.).
Remarks: This is an English translation of Rahn's Teutsche Algebra, 1659, (U.M.).
74. Author: Pierre de La Ramée.
Title: P. Rami Arithmeticae libri duo: Geometriae septem et viginti a Lazaro Schomero recogniti & aucti.
Description: t.-p., 8 p. l., 240, 178, (4), 1 l., illus., fold. tab., diagrs., 24½ cm., Frankfurt, 1599, (U.M.).
Remarks: 1st. ed., Basel, 1569, (G.A.P.); other editions of the 16th. cent.: 1577, (N.Y.P.); 1580, (G.A.P.); 1581, (N.Y.P.); 1586, (G.A.P.); 1591, 1592, (C.U.); 1596, (G.A.P.); 1599, (Schonerus), (G.A.P.); 1599, Frankfurt, 1627, (N.Y.P.); and many others up to 1636. D. E. Smith in his Rara Arithmetica also mentions an English ed. of The Art of Arithmetick in whole numbers and fractions.... by P. Ramus.....translated by William Kempe, London, 1592. There is also an Arithmetices et algebrae libri II a L. Schonero emend., Frankfurt, 1586, (G.A.P.).

75. Author: Same as preceding page.
 Title: P. Rami Arithmeticae libri duo cum commentariis
 Wilebrordi Snelli R. F.
 Description: 2 p. l., 106 pp., 19 $\frac{1}{2}$ cm., Leyden, 1613,
 (U.M.).
 Remarks: This comprises Ramus' 2 books of arithmetic
 with Snell's addition De re nummaria, 1635. Also
Arithmeticae libri tres, 1st. ed., 1555, (G.A.P.).
76. Author: Ulricus Regius.
 Title: Utriusque Arithmetices Epitome....
 Description: t.-p., 104 numb. l., diagrs., 15 $\frac{1}{2}$ cm.,
 Freiburg, 1543, (U.M.).
 Remarks: Other editions: 1536, 1550, (G.A.P.).
77. Author: Gregorius Reisch.
 Title: Margarita philosophica.
 Description: 2 ff. blank + 289 unnumb., 45 ll., 15.6
 cm., Freiburg, 1503, (G.A.P.).
 Remarks: Further editions: Strasburg, 1504, (G.A.P.);
 Freiburg, 1504, ed. s. l., 1504; Strasburg, 1508;
 Basel, 1508; Basel, 1512; Strasburg, 1512; ib.,
 1515; Basel, 1517; Paris, 1523; Basel, 1535; ib.,
 1583, (U.M.); Venice, 1594; ib., 1599; ib., 1600.
78. Author: Carlo Renaldini.
 Title: Caroli Renaldinii.....Ars analytica mathematicum
pars prima,...
 Description: 1 p. l., port., t.-p., (2), 534 pp., (33),
 diagrs., 35 cm., Florence, 1665, (U.M.).
 Remarks: There is an earlier ed., Anconnae, 1644, and
 a later one, Venice, 1684, (N.Y.P.). Riccardi lists
 7 works of Renaldini.
79. Author: Same as above.
 Title: Same as above, but pars secunda.
 Description: t.-p., 248 pp., (15), 35 cm., Patavii,
 1669, (U.M.).
80. Author: Joachim Ringelbergius.
 Title: Joachimi Fortii Ringelbergii.....Opera,.....
 Description: 687 pp. numb. + 30 = 717 pp., 22-29 ll.,
 15.7 cm., Leyden. 1531, (G.A.P.).
 Remarks: Another Leyden ed., 1539; Basel, 1541; Leyden,
 1556, (U.M.).

81. Author: Christoff Rudolff.

Title: Künstliche rechnung mit der ziffer uñ mit den
zal pfennigē sampt der Wellischen practica....

Description: t.-p., (238) pp., 1 diag., 15½ cm.,
Wien, 1532, (U.M.). Colophon, Nürnberg.

Remarks: This is an extension of the 1st. part of the
Coss, published in 1525. The book is divided into
3 parts, Das grundbuchlin, Das Regelbüchlein, Das
exempelbüchlein. There were many editions:
1st. ed., Vienna, 1526; other editions, 1532, (U.M.),
1534, 1537, 1540, (U.M.), 1546, 1553, (G.A.P.),
1557, 1561, 1574, 1588. Rudolff also published a
collection of problems in several editions.

82. Author: Pierre Savonne.

Title: L'arithmetique de Pierre Savonne, dict Talon,..

Description: t.-p., 8 p. l., 146 numb. l., (4) pp.,
17 cm., Lyons, 1571, (U.M.).

Remarks: 1st. ed., Paris, 1563; ib., 1565; Lyons,
1585; ib., 1588; ib., 1619, (U.M.), and as late as
1672.

83. Author: Johann Scheubel.

Title: Compendium arithmeticae artis,.....Iam denuò
ab ipso autore recognitum & emendatum.

Description: t.-p., 1 p. l., 193 pp., (11), 1 l.,
illus., 16 cm., Basel, 1560, (U.M.). Defective copy.
All between t.-p. & p. 15 lacking.

Remarks: 1st. ed., 1549, (G.A.P.). Scheubel also
wrote De numeris, 1545, (U.M.), Algebrae compendiosa,
1551, (U.M.), and Euclidis, 1550.

84. Author: Johan van Sesen.

Title: Flores Arithmetici edder Arithmetica (in 2 vs.).

Description: t.-p., 1069 pp. in the 2 volumes together,
19 cm., Hamburg, 1622, (N.Y.P.).

85. Author: Giovanni Sfortunati.

Title: Nuouo Lumo, Libro di Arithmetica. Institutio
Nuouo lume.....

Description: t.-p., 129 numb. l., tables, diagrs.,
21 cm., Venice, 1561, (U.M.).

Remarks: 1st. ed., Venice, 1634, (G.A.P.), very rare,
not mentioned by Riccardi. Other editions: 1543,
1544, (G.A.P.), 1545, (G.A.P.), 1550, 1568.

86. Author: Moritz Steinmetz.
Title: *Arithmeticae praecepta*.....
Description: t.-p., (236) pp. incl. tables, diagrs.,
15½ x 9 cm., Leipsig, 1575, (U.M.). At the end of
the dedication is the date 1568.
Remarks: 1st. ed., 1568, (N.Y.P.). Steinmetz also
published Euclides, 1577, (U.M.).
87. Author: Bernaert Stockmans.
Title:.....*Arithmetica*.....
Description: 4 ff. blank + 211 unnumb., 31-33 ll.,
14 cm., Dordrecht, 1609, (G.A.P.).
Remarks: 1st. ed., 1589. Of the number of editions
after 1600, the library of G. A. Plimpton has the
following: 1637, 1644, 1653, 1676. There is also
the ed., Utrecht, 1633, (U.M.).
88. Author: Siegmund Suevus.
Title: *Arithmetica historica. Die Lößliche Rechen-
kunst*....
Description: t.-p., 8 p. l., 455 pp., (50), 1 l.,
19 x 15½ cm., Breslau, 1593, (U.M.).
89. Author: Girolamo Tagliente.
Title: *Libro da abaco*.....*Restampata nouamente del*
1520.
Description: t.-p., (157) pp., 1 l. ill. (incl. tables,
diagrs.), 15 cm., Venice, 1520, (2nd. ed.), (U.M.).
Remarks: Smith, *Rara Arithmetica* (1908), v. I, pp. 114,
115 lists 36 editions; 1st. ed., Venice, 1515,
(G.A.P.). Giovanni Antonio Tagliente is joint
author of above work.
90. Author: Niccolò Tartaglia.
Title: *La prima parte del general trattato di numeri,
et misure di Nicolo Tartaglia*.....
Description: t.-p., (8), 277 numb. l., illus., ports.,
tables, diagrs., 30 cm., Venice, 1556, (U.M.).
Remarks: There were subsequent editions of this work
in 1560, 1592-1593, 1578 (Paris), (D.E.S.), 1578
(Antwerp). He also published Tutte l'opera arith-
metica, 1592, and L'arithmetique, Paris, 1613,
(N.Y.P.), besides other works on Euclid's Elements,
etc.

91. Author: Cuthbert Tunstall.
Title: De arte supputandi libri quatuor, Cutheberti Tunstalli.....
Description: t.-p., 259 pp., 20½ cm., Paris, 1538, (U.M.).
Remarks: 1st. ed., London, 1522, (G.A.P.); other editions: 1529, (G.A.P.), (D.E.S.); 1535, 1543, 1544, (Strasburg), (U.M.), 1548, 1551.
92. Author: Theodor Tzwivel.
Title: Arithmetice oposcula duo Theodorici tzwivel....
Description: 10 ff., 46 ll., 20 cm., Cologne, 1507, (G.A.P.).
Remarks: There is also a Cologne ed. in 1505.
93. Author: Josephus Unicornio.
Title: De l'arithmetica universale del Sig. Joseppo Unicornio....
Description: t.-p., 2 v. in 1, 8 p. l., 395 numb. l., (1) p., 22 cm., Venice, 1598, (U.M.).
Remarks: He also wrote De utilitate mathematicarum artium, Venice, (G.A.P.), (D.E.S.).
94. Author: Giacomo Venturoli.
Title: Breve compendiodi tutte le regole dell'arithmetica pratica..... Seconda Impressione.
Description: t.-p., 6 p. l., 208 pp., 140 diagrs. on 9 pl., 20 cm., Bologna, 1681, (U.M.).
Remarks: This says 2nd. ed. The 1st. ed. was published in 1663 with title: Ordini aritmetici..... There are also 3d. and 4th. ed., 1705 and 1754, (U.M.). The Dialogo aritmetico was first published separately in 1666 with title: Scorta di economia, ò sia dialogo aritmetico. (cf. Riccardi. Bibl. mat. ital.). Riccardi records 1666 and 1672 editions, but gives no date for the Bologna ed. just mentioned. We also have the 4th. ed., Bologna, 1745, (U.M.).
95. Author: Francois Viète.
Title: L'algebre nouvelle de Mr. Viete..... Traduite en Francois Par A. Vasset.
Description: t.-p., 15 p. l., 186 pp., (2), diagrs., 21½ cm., Paris, 1630, (U.M.).
Remarks: This is a French translation of In artem analyticam isagoge, Turonis, 1591, and of Zeticorum

libri quinque, 1593. Vieta also wrote on works of Apollonius. There is also Hume's L'algèbre de Viète, Paris, 1636, (U.M.).

96. Author: Same as above.

Title: Francisci Vietae Opera mathematica, in unum Volumen congesta, ac recognita, Operâ atque studio Francisci à Schooten....

Description: t.-p., 6 p. l., 554 pp., illus., diags., 31 cm., Lugduni Batavorum, 1646, (U.M.).

97. Author: Nicolaus Voigtel.

Title: Geometria subterranea, oder Marckscheide-kunst(No imprint).

Description: t.-p., and half-title, 6 p. l., 152 pp., incl. tables, illus., 9 fold. pl., diags., 32 cm., Eisleben (?), 1686, (U.M.).

Remarks: Murhard lists editions corresponding to this as follows: Eisleben, Gedruckt Johann Dietzeln, 1686; 1688; Leipsig, 1692. Date in ms. on t.-p.: 1691. We also have a later ed., Eisleben, 1713, (U.M.).

98. Author: Henricus Vuelpius.

Title: Libellus de communibus et usitatis arithmeticae practicae regulis,.....

Description: t.-p., (115) pp., illus., diags., 19½ cm., Cologne, 1544, (U.M.).

99. Author: Caspar Waser.

Title: Institutio arithmetica brevis & facilis.....

Description: t.-p., 92 numb. l., illus., II fold. pl., 16 cm., Tiguri, 1603, (U.M.).

Remarks: Waser also wrote geometrical works.

100. Author: Johannes Wolphius.

Title: Rudimenta arithmetices.....

Description: t.-p., (111) pp., diags., 14 cm., Colophon: Frankfurt, 1534, (U.M.).

Remarks: 1st. ed., Nürnberg, 1527; other Frankfurt editions: 1537, 1548, (G.A.P.), 1561; Strasburg editions: 1539, 1540. This arithmetic is followed by geometry by Johannes Vogelin.

101. Author: Giovanni Battista Zuchetta.
Title: Prima parte della Arimmetica di Gio. Battista Zuchetta....
Description: t.-p., 1 p. l., (24), 412 pp., (4), (10) pp. incl. tables, diagrs., 36 cm., Brescia, 1600, (U.M.). 10 pp. of ms. tables at end of book.

References to works are given by numbers. When no date is added, reference is made to the particular edition described.

To indicate the library in which are found the books examined, the following abbreviations are used:

Library of the University of Michigan, (U.M.).
Library of George A. Plimpton, New York City, (G.A.P.).
Library of David Eugene Smith, Teachers College,
Columbia University, (D.E.S.).
Library of Columbia University, (C.U.).
New York Public Library, (N.Y.P.).

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